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Cartes de grand genre : de la hiérarchie KP aux limites probabilistes

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Résumé

Cette thèse s'intéresse aux *cartes combinatoires*, qui sont définies comme des plongements de graphes sur des surfaces, ou de manière équivalente comme des recollements de polygones. Le *genre* g de la carte est défini comme le nombre d'anses que possède la surface sur laquelle elle est plongée.

En plus d'être des objets combinatoires, les cartes peuvent être représentées comme des factorisations de permutations, ce qui en fait également des objets algébriques, qu'on peut notamment étudier grâce à la théorie des représentations du groupe symétrique. En particulier, ces propriétés algébriques des cartes font que leur série génératrice satisfait la *hiérarchie KP* (et sa généralisation, la *hiérarchie 2-Toda*). La hiérarchie KP est un ensemble infini d'équations aux dérivées partielles en une infinité de variables. Les équations aux dérivées partielles de la hiérarchie KP se traduisent ensuite en formules de récurrence qui permettent d'énumérer les cartes en tout genre.

D'autre part, il est intéressant d'étudier les propriétés géométriques des cartes, et en particulier des très grandes cartes aléatoires. De nombreux travaux ont permis d'étudier les propriétés géométriques des cartes *planaires*, c'est à dire de genre 0. Dans cette thèse, on étudie les *cartes de grand genre*, c'est à dire dont le genre tend vers l'infini en même temps que la taille de la carte. Ce qui nous intéressera particulièrement est la notion de *limite locale*, qui décrit la loi du voisinage d'un point particulier (la *racine*) des grandes cartes aléatoires uniformes.

La première partie de cette thèse (Chapitres 1 à 3) est une introduction à toutes les notions nécessaires : les cartes, bien entendu, mais également la hiérarchie KP et les limites locales. Dans un deuxième temps (Chapitres 4 et 5), on cherchera à approfondir la relation entre cartes et hiérarchie KP, soit en expliquant des formules existantes par des constructions combinatoires, soit en découvrant de nouvelles formules. La troisième partie (Chapitres 6 et 7) se concentre sur l'étude des limites locales des cartes de grand genre, en s'aidant notamment de résultats obtenus grâce à la hiérarchie KP. Enfin le manuscrit se conclut par quelques problèmes ouverts (Chapitre 8).

Plus précisément, on trouvera dans le chapitre 1 des définitions détaillées

des cartes sous tous les angles, ainsi qu'un aperçu historique de l'étude des cartes. Le chapitre se termine par un exposé des résultats obtenus ainsi qu'un plan du manuscrit.

Le chapitre 2 est une boîte à outils permettant de construire les hiérarchies KP et 2-Toda de manière algébrique, à partir d'objets simples comme les partitions d'entiers et les fonctions symétriques, grâce à des résultats de théorie des représentations. On y trouve la preuve que la série génératrice des cartes est une solution de ces hiérarchies.

Le chapitre 3 est également une boîte à outils : il approfondit la notion de limite locale présentée dans l'introduction. On y présente des cartes infinies du plan qui ont des propriétés hyperboliques et qui sont les limites locales des cartes de grand genre, ainsi qu'une méthode d'exploration pas-à-pas des cartes infinies : le *processus de peeling*.

Dans le chapitre 4, on donne une explication *bijective* de certaines formules issues de la hiérarchie KP, comme la formule de Goulden–Jackson, dans le cas des cartes planaires. Ces formules avaient été obtenues dans un premier temps par des méthodes algébriques, on en donne ici une preuve bijective, c'est à dire une correspondance explicite et combinatoire entre les cartes elles-mêmes. Cela nous permet d'obtenir des formules plus précises, et on termine par un premier petit pas vers une bijection unifiée pour les cartes en tout genre.

Dans le chapitre 5, on découvre de nouvelles formules pour les cartes, grâce à la hiérarchie 2-Toda. Ces formules, valables en tout genre, concernent un vaste ensemble de cartes, notamment les cartes biparties à degrés prescrits.

Dans le chapitre 6, on résout la conjecture de Benjamini–Curien, qui postule que les *triangulations* uniformes de grand genre convergent localement vers des triangulation infinies du plan. La preuve utilise à la fois des arguments combinatoires et des arguments probabilistes, en particulier elle s'appuie sur la formule de récurrence de Goulden–Jackson. La convergence locale a pour corollaire un résultat d'énumération asymptotique des triangulations de grand genre.

Dans le chapitre 7, on étend les résultats du chapitre précédent aux cartes biparties à degré prescrits.qui forment une famille de cartes à une infinité de paramètres. Nous obtenons ainsi un résultat *d'universalité* : de nombreux modèles de cartes présentent un comportement analogue. La preuve est similaire à celle du chapitre 6, quoique plus complexe puisque le modèle étudié est beaucoup plus général. On utilise en particulier les résultats du chapitre 5.

Pour conclure, le chapitre 8 présente quelques problèmes ouverts motivés par le travail de cette thèse. On y trouvera des problèmes bijectifs, des questions plus générales sur la hiérarchie KP et les cartes, et des questions variées sur la géométrie des cartes de grand genre.

Short summary

This thesis focuses on combinatorial maps, which are defined as embeddings of graphs on surfaces, or equivalently as gluing of polygons. The genus g of the map is defined as the number of handles of the surface on which it is embedded. In addition to being combinatorial objects, the maps can be represented as factorizations of permutations, which also makes them algebraic objects, which one can study in particular thanks to the representation theory of the symmetric group. In particular, these algebraic properties of maps mean that their generating series satisfies the KP hierarchy (and its generalization, the 2-Toda hierarchy). The KP hierarchy is an infinite set of partial differential equations in an infinity of variables. The partial differential equations of the KP hierarchy are then translated into recurrence formulas which make it possible to enumerate maps of any genus. On the other hand, it is interesting to study the geometric properties of maps, and in particular very large random maps. Many works have focused on the geometrical properties of planar maps, i.e. of genus 0. In this thesis, we study maps of large genus, that is to say whose genus tends towards infinity at the same time as the size of the map. What will particularly interest us is the notion of local limit, which describes the law of the neighborhood of a particular point (the root) of large uniform random maps. The first part of this thesis (Chapters 1 to 3) is an introduction to all the necessary concepts: maps, of course, but also the KP hierarchy and local limits. In a second part (Chapters 4 and 5), we will seek to deepen the relationship between maps and the KP hierarchy, either by explaining existing formulas by combinatorial constructions, or by discovering new formulas. The third part (Chapters 6 and 7) focuses on the study of the local limits of large maps, using in particular the results obtained from the KP hierarchy. Finally the manuscript ends with some open problems (Chapter 8).

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[1] liste non exhaustive.

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Chapitre 1

Introduction

La géométrie — que l'on peut définir très vaguement comme « l'étude des formes et des longueurs » — est un domaine qui occupe une place centrale dans l'histoire des sciences, que ce soit en mathématiques, en physique, ou en informatique. Motivée au départ par des considérations d'ordre pratique (géométrie du plan et des volumes), elle s'est progressivement diversifiée et a étendu son influence à de très nombreux domaines.

Au milieu de ces vastes territoires, cette thèse a pour ambition d'être une minuscule contribution à la géométrie, plus précisément à la géométrie discrète. Elle s'inscrit en premier lieu dans le domaine de la combinatoire, mais elle emprunte aussi aux probabilités, à l'algèbre et même à la physique mathématique. Notre attention se portera plus spécifiquement (et quasi exclusivement) sur les structures discrètes que sont les *cartes combinatoires*. Il s'agit d'objets formidables, aux multiples définitions (dessin de graphes, patchwork de polygones, factorisation de permutations), mais surtout que l'on rencontre dans des endroits très variés. On les retrouve évidemment dans les domaines mentionnés ci-dessus, ceux que l'on abordera dans cette thèse, mais leur champ d'action est beaucoup plus vaste et on fait usage des cartes jusqu'en géométrie algébrique et en informatique graphique.

On va maintenant présenter les cartes d'un peu plus près, et s'y intéresser en particulier du point de vue de la combinatoire bijective, algébrique et probabiliste.

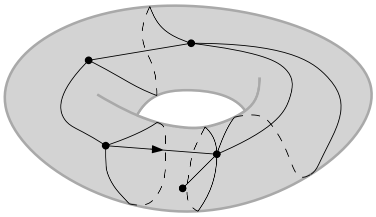


FIGURE 1.1 – Une carte de genre 1.

1.1 Les cartes combinatoires

1.1.1 Comme graphes plongés

Une carte est le plongement d'un graphe sur une surface (voir Figure 1.1). Bien évidemment cette définition est vague — voire carrément fausse — à moins d'adopter des définitions assez précises des mots graphe, plongement, et surface.

Un *graphe* est la donnée d'un ensemble de *sommets*, et d'un ensemble d'*arêtes*, chaque arête reliant deux sommets. On s'autorise ici à ce qu'une arête relie un sommet à lui-même (on parlera de *boucle*), ou bien que plusieurs arêtes relient la même paire de sommets (on parlera parfois d'*arêtes multiples*) — pour désigner ce genre de graphes on parle parfois de multigraphe. Par la suite, sauf mention contraire, nous nous restreindrons aux graphes *connexes* (c'est à dire qu'entre toute paire de points il existe un chemin d'arêtes dans le graphe) ayant un nombre fini d'arêtes.

Une *surface* sera ici une surface[1] compacte connexe et orientée[2]. D'après le théorème de classification des surfaces (voir par exemple [MT01]), à homéomorphisme près, chaque surface est décrite par un unique entier $g \geqslant 0$, qu'on appellera son *genre*. La surface de genre 0 est la sphère, la surface de genre 1 est le tore, et la surface de genre g est la somme connexe (en quelque sorte, le recollement) de g tores.

Un *plongement* est le dessin sans croisement d'un graphe sur une surface, et sera toujours compris à homéomorphisme préservant l'orientation près. De plus, on impose la condition que pour que le plongement d'un graphe G sur une surface S soit valide, il faut que $S \setminus G$ soit homéomorphe à une union disjointe de disques ouverts — on parle de plongement cellulaire.

Les arêtes et sommets de G constituent les arêtes et sommets de la carte, et les *faces* de la carte sont les composantes connexes de $S \setminus G$. Un *coin* est un

[1] c'est à dire une variété différentielle de dimension 2.

[2] il existe des travaux sur les « cartes non orientables », mais on n'en parlera pas dans cette thèse.

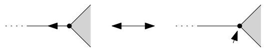


FIGURE 1.2 – Passer d'une arête orientée à un coin marqué, et vice-versa.

secteur angulaire entre deux demi-arêtes consécutives autour d'un sommet. Pour des raisons de symétrie, on considèrera toujours des cartes enracinées, c'est à dire munies d'une arête distinguée et orientée qu'on appellera la *racine*. Le fait d'enraciner une carte brise toutes les symétries possibles : pour chaque partie de la carte (sommet, arête, face, coin), on peut donner un « itinéraire » vers cet endroit depuis la racine. Le sommet au départ de la racine est appelé sommet racine, et la face à gauche de la racine est appelée face racine. De manière équivalente, on peut décider d'enraciner une carte non pas sur une arête mais sur un coin (il existe une bijection entre les arêtes orientées et les coins de la carte, voir Figure 1.2). Dans ce cas il sera souvent commode de considérer que la racine sépare le coin d'origine en deux nouveaux coins distincts, l'un à gauche de la racine, l'autre à sa droite.

Le *genre* de la carte est le genre de S. Une carte de genre 0 est aussi appelée *carte planaire*. En effet, on peut décider de passer de la sphère au plan par projection stéréographique à partir d'un point quelconque de la face racine, qu'on appelle alors également *face externe* ou *face infinie*.

Le genre g, le nombre d'arêtes n, de sommets v et de faces f sont reliés par la *formule d'Euler* :

$$v - n + f = 2 - 2g.$$

Le *degré* d'un sommet (resp. d'une face) est égal aux nombres de coins qui lui sont incidents.

Le *dual* d'une carte est obtenu par l'opération suivante (voir Figure 1.3) : on place un sommet sur chaque face, et pour chaque arête e adjacente à deux faces f_1 et f_2, on trace une arête e^* entre les sommets f_1^* et f_2^* correspondants à f_1 et f_2. La racine de la carte duale est l'arête qui croise la racine de la carte de départ (on parle aussi de carte primale), orientée de telle sorte que le sommet racine de la carte duale soit sur la face racine de la carte primale. L'opération de dualité est une involution (à retournement de la racine près).

1.1.2 Comme recollements de polygones

De manière totalement équivalente, on peut voir les cartes comme des recollements de polygones. On se réfèrera à [MT01] pour une exposition détaillée de l'équivalence entre les présentations combinatoire et topologique des cartes.

Dans ce qui suit un *polygone* sera une surface à bord orientée, homéomorphe à un disque, ayant sur son bord un certain nombre de sommets,

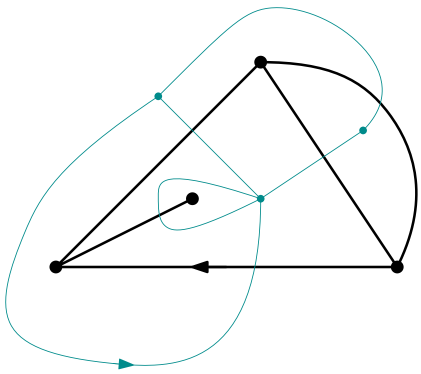


FIGURE 1.3 – Une carte (en noir, gras) et sa carte duale (en bleu, fin).

deux sommets consécutifs étant reliés par une arête (de manière équivalente, le bord d'un polygone est l'union disjointe de sommets et d'arêtes). L'intérieur du polygone est appelé une face.

Remarque 1.1.1. *Les polygones définis ici sont des objets abstraits, ils n'ont qu'une face, il n'y a rien de l'autre côté.*

On s'autorise à recoller deux polygones selon leurs arêtes en préservant l'orientation (voir Figure 1.4). Les arêtes respectives sont alors fusionnées, et il en va de même pour les sommets aux extrémités de l'arête.

On peut alors donner une définition alternative (et équivalente) d'une carte combinatoire comme la donnée d'un ensemble de polygones qu'on a recollés pour former une surface (connexe). Dans cette construction, il est naturel de considérer également des cartes sur des surfaces non-nécessairement connexes, c'est-à-dire composées de plusieurs composantes connexes.

1.1.3 Cartes étiquetées et factorisations de permutations

On va maintenant définir des objets proches, les cartes étiquetées, avant de les relier aux cartes enracinées. Il est plus pratique cette fois de les définir sans la contrainte de connexité.

Une *carte étiquetée* (voir Figure 1.5) est une carte (non enracinée, non-nécessairement connexe) telle que chaque coin porte une étiquette distincte. Plus précisément, soit M une carte à n arêtes, elle possède $2n$ coins qui portent chacun une étiquette distincte entre 1 et $2n$. On peut définir trois permutations σ, ϕ et α de la manière suivante :

- chaque cycle de σ correspond à un sommet, il encode l'ordre des coins autour de ce sommet dans le sens horaire,

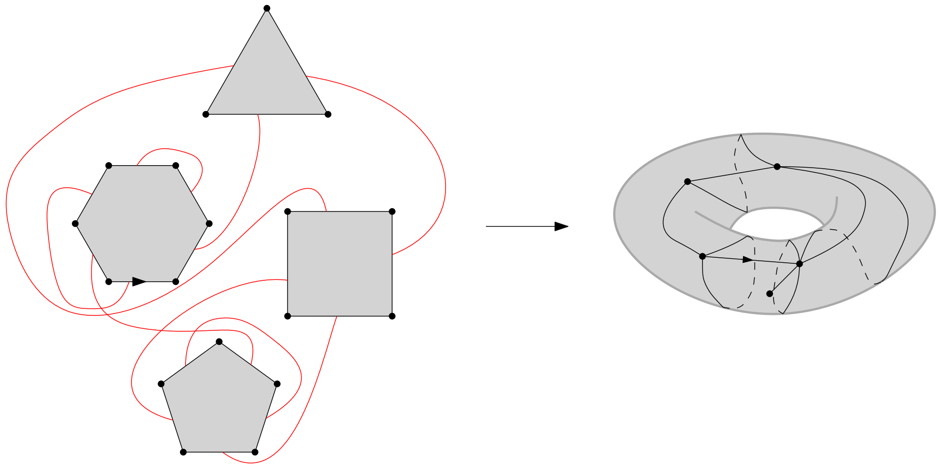
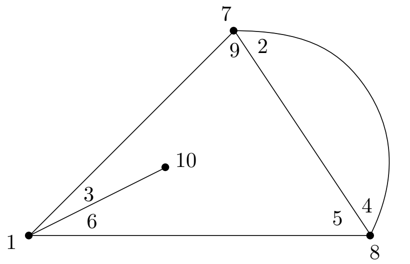


FIGURE 1.4 – Une carte comme recollement de polygones.



$$\sigma = (1,3,6)(2,9,7)(4,8,5)(10)$$
$$\phi = (1,8,7)(2,4)(3,9,5,6,10)$$
$$\alpha = (1,9)(2,5)(3,10)(4,7)(6,8)$$

FIGURE 1.5 – Une carte étiquetée et ses permutations associées, représentées sous forme de produit de cycles.

- chaque cycle de ϕ correspond à une face, il encode l'ordre des coins autour de cette face dans le sens horaire,
- α est une involution sans point fixe, chaque 2-cycle correspond à une arête. Plus précisément, l'arête correspondant au 2-cycle (i,j) est « adjacente à gauche » aux coins étiquetés i et j (voir Figure 1.6 gauche).

On a alors, en multipliant les permutations de droite à gauche (voir Figure 1.6 droite) :

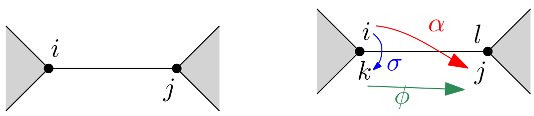


FIGURE 1.6 – Gauche : le 2-cycle associé à cette arête est (i,j). Droite : les permutations σ, ϕ et α et leur action.

$$\phi\sigma = \alpha.$$

On rappelle que le *type cyclique* d'une permutation est le multiensemble des tailles de ses cycles.

Proposition 1.1.2. *Il y a une bijection entre les cartes étiquetées à n arêtes et les triplets de permutations (σ, ϕ, α) de $\mathfrak{S}_{2n}$ tels que α est une involution sans points fixes et $\phi\sigma = \alpha$.*

Le type cyclique de σ (resp. ϕ) correspond dans cette bijection à la distribution des degrés des sommets (resp. faces) de la carte associée.

La carte étiquetée est connexe si et seulement si $\langle\sigma, \phi\rangle$, le groupe engendré par σ et ϕ, agit transitivement sur $\{1, 2, \ldots, 2n\}$, c'est)-à-dire que pour toute paire (i, j), il existe $\gamma \in \langle\sigma, \phi\rangle$ telle que $\gamma(i) = j$.

Si on se restreint aux objets connexes, il existe une correspondance claire entre cartes enracinées et cartes étiquetées.

Lemme 1.1.3. *Soit $\mathcal{M}_n$ l'ensemble des cartes enracinées (connexes) à n arêtes, et $\mathcal{M}_n^*$ l'ensemble des cartes étiquetées connexes à n arêtes. Il existe une opération « $(2n-1)!$-to-1 » de $\mathcal{M}_n^*$ vers $\mathcal{M}_n$.*

Démonstration. Soit une carte de $\mathcal{M}_n^*$. On transforme le coin numéro $2n$ en coin racine, et on efface les autres étiquettes. Comme l'enracinement brise toutes les symétries possibles, la donnée des autres étiquettes est simplement une permutation de $\mathfrak{S}_{2n-1}$. □

L'introduction des cartes étiquetées permet d'étudier des objets en apparence purement combinatoires et géométriques d'un point de vue algébrique, et plus précisément grâce à la théorie des représentations qui nous fournit des outils très puissants (voir Section 1.2.3).

1.1.4 Modèles particuliers de cartes

Maintenant que l'on a défini (et redéfini) les cartes en général, on peut introduire ici des modèles particuliers de cartes qui nous intéresseront particulièrement dans cette thèse. En particulier, on définira les constellations qui sont le modèle de cartes le plus général et que nous utiliserons dans la suite.

Une *triangulation* est une carte dont toutes les faces ont degré 3. De même, une *quadrangulation* est une carte dont toutes les faces ont degré 4.

Une *carte à bord(s)* est une carte dont certaines faces sont distinguées, qu'on appelle des *bords*. Un bord est dit *simple* s'il est incident à autant de sommets que de coins. Selon les contextes, une carte à bord sera soit

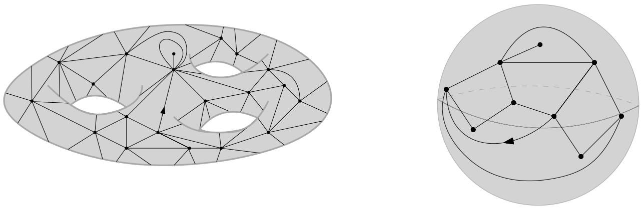


FIGURE 1.7 – Une triangulation et une quadrangulation.

enracinée sur une arête quelconque, soit multi-enracinée, avec une racine sur chaque bord.

Une *carte bipartie* est une carte avec la condition supplémentaire que chaque sommet est soit blanc soit noir, et que chaque arête relie un sommet blanc à un sommet noir. Par convention, le sommet racine est blanc.

Une r-*constellation* (pour $r \geqslant 2$, voir Figure 1.8 gauche) est une carte vérifiant les contraintes suivantes :

- il existe deux types de sommets : des *sommets étoiles*, et des *sommets coloriés*,
- chaque arête relie un sommet colorié à un sommet étoilé,
- chaque sommet colorié porte une couleur qui est un entier de $[1, r]$,
- les sommets étoiles ont tous degré r, et sont adjacents à un sommet de couleur 1, puis un sommet de couleur 2, ..., puis un sommet de couleur r (dans cet ordre cyclique antihoraire),
- une constellation est dite enracinée si elle possède un sommet étoile distingué.

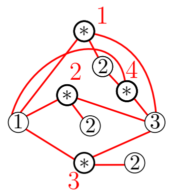
Les 2-constellations sont exactement les cartes biparties : partant d'une carte bipartie, il suffit de rajouter un sommet étoile au milieu de chaque arête pour obtenir une 2-constellation.

Une *constellation étiquetée* à n sommets étoiles est une constellation (non enracinée, non-nécessairement connexe) dont chaque sommet étoile porte une étiquette distincte dans $[1, n]$.

Proposition 1.1.4. *Il existe une bijection entre les r-constellations étiquetées C et les $(r+1)$-uplets de permutations $(\sigma_1, \sigma_2, \ldots, \sigma_r, \phi)$ vérifiant*

$$\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_r = \phi.$$

Pour tout i, la permutation σ_i encode les sommets de couleur i, et la distribution des degrés des sommets de couleur i de C correspond au type cyclique



$$\sigma_1 = (1,2,3,4)$$
$$\sigma_2 = (1,4)(2)(3)$$
$$\sigma_3 = (1,3,2,4)$$
$$\phi = (1,4)(2)(3)$$

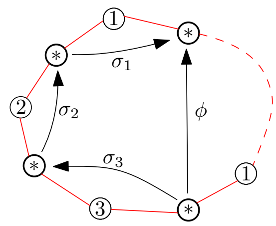


Figure 1.8 – Gauche : une 3-constellation et ses permutations associées. Droite : la permutation ϕ qui décrit les faces.

de σ_i. La permutation ϕ encode les faces. Toutes les faces de C ont un degré multiple de r, le pseudo-degré *d'une face est égal au nombre de coins incidents à cette face qui sont incidents à un sommet de couleur* 1*. Alors la distribution des pseudo-degrés des faces de C correspond au type cyclique de ϕ (voir Figure 1.8).*

De plus, il existe une correspondance $(n-1)!$-to-1 entre les constellations étiquetées connexes à n sommets étoiles et les constellations enracinées (connexes) à n sommets étoiles.

Il est classique de voir les cartes générales comme des cas particuliers de cartes biparties (et donc de 2-constellations) où tous les sommets blancs ont degré 2, on donne ici une version un peu différente de cette correspondance :

Proposition 1.1.5. *Les cartes à n arêtes sont en bijection avec les quadrangulations biparties à n faces.*

Démonstration. Cela se voit facilement sur les cartes étiquetées : partant de la présentation en factorisations de la Proposition 1.1.4, on prend $r = 2$, et on contraint ϕ à être une permutation sans points fixes. D'une part, on retrouve la présentation des cartes en factorisations de permutations de la Proposition 1.1.2, d'autre part on a une 2-constellation, c'est-à-dire une carte bipartie, qui n'a que des faces de pseudo-degré 2, donc de degré 4.

Cette bijection peut également se décrire directement sur les cartes (voir Figure 1.9 pour un exemple). Partant d'une carte M, dans chaque face f de M, dessiner un sommet blanc f^*. Pour chaque coin de M, relier par une arête le sommet (noir) auquel il est incident au sommet blanc correspondant à la face à laquelle il appartient. Oublier les arêtes de la carte de départ. Cette construction est due à Tutte [Tut63].

□

Dans le cas planaire, les quadrangulations et les quadrangulations biparties sont en bijection. Dans un sens, on peut oublier les couleurs des sommets,

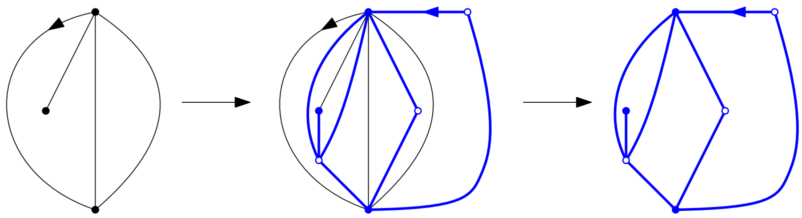


FIGURE 1.9 – La bijection entre cartes (en noir, fin) et quadrangulations biparties (bleu, gras).

dans l'autre, tout cycle simple borde un ensemble de faces[3] et est donc de longueur paire, donc une quadrangulation planaire est forcément bipartie.

1.1.5 Cartes infinies du plan

Comme dit précédemment, la plupart des cartes étudiées dans cette thèse sont finies, à une exception (notable) près : les *cartes infinies du plan*. Une carte infinie du plan est le plongement (sans point d'accumulation) d'un graphe infini dans le plan, de telle sorte que chaque sommet et chaque face aient degré fini[4] (par exemple : la grille $\mathbb{Z}^2$). Ces cartes sont ici encore enracinées. Ces objets vont nous permettre par la suite d'étudier des limites de cartes.

La *boule* de rayon r d'une carte m (finie ou infinie du plan) est la carte à bord formée des faces de m adjacentes à un sommet à distance au plus $r-1$ de la racine, ainsi que toutes leurs arêtes et tous leurs sommets. On notera cette boule $B_r(m)$. Par convention, $B_0(m)$ contient uniquement le sommet racine.

Remarque 1.1.6. *Cette définition est un peu différente de celle des boules sur les graphes, mais elle est plus pratique pour l'étude des limites locales (voir Section 1.2.4).*

La *distance locale* (voir Figure 1.10) entre deux cartes m et m' est :

$$d_{loc}(m, m') = (1 + \max\{r \geqslant 0 | B_r(m) = B_r(m')\})^{-1} .$$

On peut vérifier qu'il s'agit bien d'une distance sur l'espace des cartes finies (et son complété, qui contient notamment les cartes infinies du plan). Cela nous permet de comparer les cartes entre elles, et même de comparer des

[3] cela est dû au fait que sur la sphère, tous les cycles sont contractibles.

[4] si on veut autoriser les sommets à avoir degré infini, il est possible de définir une carte infinie du plan comme recollement de polygones.

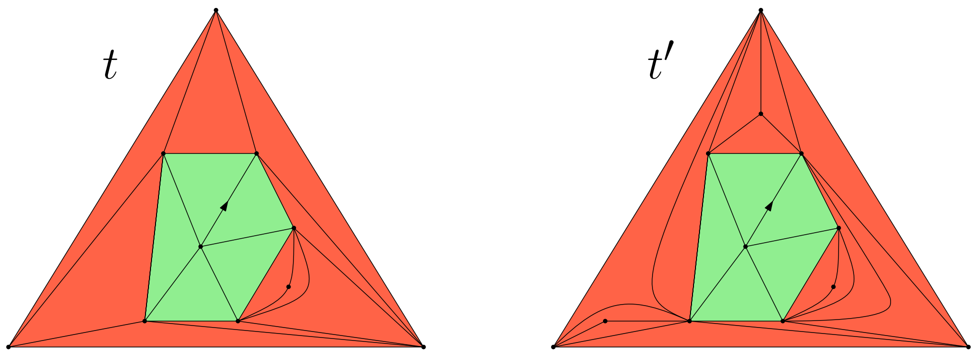


FIGURE 1.10 – Dans les deux triangulations, les boules de rayon 1 coincident mais pas celles de rayon 2, on a donc $d_{loc}(t, t') = 1/2$.

cartes finies aux cartes infinies du plan : une très grande carte finie peut en ce sens « ressembler » à une carte infinie. C'est ce qui va nous permettre de voir les cartes infinies du plan comme des limites de cartes finies (par exemple, la grille $[-N, N]^2$ enracinée en $(0, 0)$ tend vers $\mathbb{Z}^2$ pour cette topologie).

☞ À partir de maintenant, toutes les cartes que l'on considérera seront connexes et enracinées, sauf mention contraire (en particulier lorsqu'on parlera de cartes étiquetées).

1.2 Un bref historique de l'étude des cartes

1.2.1 L'énumération des cartes

Un des axes principaux de la combinatoire est l'énumération, et les cartes, en tant que structures combinatoires, se prêtent bien à l'exercice. On va donc commencer notre tour d'horizon des cartes par de l'énumération, en se concentrant surtout sur l'énumération par séries génératrices, les méthodes bijective et algébrique auront leur section dédiée. Avant de commencer, il nous faut mentionner que bon nombre des résultats de cette section s'appuient sur la *méthode symbolique* et sur de l'analyse de singularités, on trouvera une bonne introduction à ces techniques dans [FS09].

On peut par exemple se poser la question suivante : « Combien y-a-t-il de cartes planaires à n arêtes ? » . C'est Tutte qui y a répondu en 1963 [Tut63] : il y en a exactement

$$\frac{2 \cdot 3^n (2n)!}{n!(n+2)!}. \tag{1.2.1}$$

Il a obtenu ce résultat alors qu'il cherchait à démontrer le théorème des 4 couleurs. Sa méthode repose sur une idée assez simple : prenons une carte, supprimons son arête racine et regardons ce qui se passe.

Plus précisément, cela nous donne une équation sur la *série génératrice* des cartes. Soit a_n le nombre de cartes planaires à n arêtes, on définit la série génératrice (formelle) des cartes planaires $A(x) = \sum_{n \geqslant 0} a_n x^n$. On peut également définir une série à deux variables : soit $a_{n,p}$ le nombre de cartes planaires à n arêtes telles que la face racine est de degré p, et alors $A(x,y) = \sum_{n,p \geqslant 0} a_{n,p} x^n y^p$. On peut retrouver la série univariée à partir de la série bivariée puisque $A(x) = A(x,1)$.

En analysant l'effet de la suppression de l'arête racine, Tutte obtient l'équation suivante :

$$A(x,y) = 1 + xy^2 A(x,y)^2 + xy\frac{A(x,y) - A(x,1)}{y-1},$$

qu'il résout astucieusement pour trouver $a_n = \frac{2 \cdot 3^n (2n)!}{n!(n+2)!}$. L'utilisation de la série bivariée est ici cruciale pour pour pouvoir écrire une équation, même si le but est de déterminer la série univariée $A(x)$: la variable y est donc appelée *variable catalytique*.

La force de la méthode de Tutte — c'est-à-dire suppression de la racine et série à une variable catalytique — est qu'elle est universelle : elle peut être appliquée à de nombreux modèles de cartes : par exemple les triangulations [Tut62], et de nombreux autres modèles à degrés contraints [Gao93], mais aussi les cartes non séparables [Bro63]. On peut également généraliser l'équation de Tutte aux cartes de genre > 0 [BC86], ou aux constellations (voir [Fan16], Chapitre 4).

Pour le genre supérieur, Bender et Canfield [BC86] ont pu démontrer des résultats asymptotiques grâce à des méthodes d'analyse de singularités : si $a_{g,n}$ compte le nombre de cartes de genre g à n arêtes, alors

$$a_{g,n} \sim t_g n^{5(g-1)/2} 12^n$$

quand g est fixé et n tend vers l'infini, où t_g est une constante[5] (la même question se pose quand g et n tendent en même temps vers l'infini, on y répond dans les chapitres 6 et 7, voir aussi Section 1.3.3).

Remarque 1.2.1. *Il n'existe pas de « formule exacte » pour $a_{g,n}$, c'est-à-dire de formule qui soit close et simple*[6] *comme* (1.2.1). *Dans cette thèse, on*

[5] les coefficients t_g peuvent se calculer grâce à une récurrence quadratique [BGR08, LZ04].

[6] la définition varie selon les personnes — quoiqu'il en soit, il n'en existe pas pour le moment selon la définition de la plupart des gens.

se concentrera sur l'énumération exacte, asymptotique, ou par formules de récurrence. Les questions de la régularité de la série génératrice à genre fixé (pour les cartes, voir par exemple [BC91]) ne seront pas traitées.

Plus récemment, il a été découvert que les cartes et la méthode de Tutte s'inscrivent dans un contexte plus large, celui de la *récurrence topologique*. La récurrence topologique [Eyn16], appliquée aux cartes, permet de calculer les séries génératrices des cartes à genre et nombre de bords fixé. Plus précisément, soit $\omega_{g,n}$ la série génératrice des cartes de genre g à n bords, alors, après changement de variable, les $\omega_{g,n}$ satisfont une récurrence quadratique qui les détermine tous si on connait $\omega_{0,1}$ et $\omega_{0,2}$: on appelle cette donnée initiale la *courbe spectrale*. La force de la récurrence topologique est qu'elle est universelle (ou presque) : pour un grand nombre de modèles topologiques (comme les volumes de Weil–Petersson ou les nombres d'intersection de Witten–Konstevich, voir [Eyn14]), les séries génératrices à genre et nombre de bords fixés vérifient la même formule, seule la courbe spectrale change.

L'énumération des cartes par méthodes analytiques sur les séries génératrices est encore aujourd'hui un outil très prolifique. Les problèmes à une variable catalytique sont désormais très bien compris grâce aux travaux de Bousquet-Mélou et Jehanne [BMJ06]. Des travaux récents s'intéressent au cas de deux variables catalytiques, par exemple [BBM17, AMS18], quand d'autres couplent bijections et méthode symbolique [BMEP20].

1.2.2 L'approche bijective

Les cartes, en tant que structures combinatoires, se prêtent particulièrement bien à une étude bijective : il s'agit de trouver des correspondances explicites entre plusieurs modèles de cartes, ou bien entre des cartes et des objets plus simples. Il est souvent plus difficile de faire de l'énumération bijective, en comparaison avec l'aspect systématique de la méthode symbolique, cependant l'intérêt de l'étude bijective est de fournir des informations fines sur la structure des objets étudiés, ce qui peut avoir des applications assez profondes, notamment dans l'étude des objets aléatoires. De plus, bien souvent l'approche bijective permet d'obtenir des formules plus précises qu'auparavant (on pourra le constater par exemple dans le chapitre 4). Enfin — c'est une opinion très personnelle, mais je sais qu'elle est partagée par d'autres — une preuve bijective d'une formule déjà établie répond de manière satisfaisante à la question vague suivante : « pourquoi cette formule est-elle vraie ? » .

Dans le monde des cartes, la bijection la plus utilisée à ce jour est la bijection de Cori–Vauquelin–Schaeffer [CV81, Sch98], qui met en correspondance

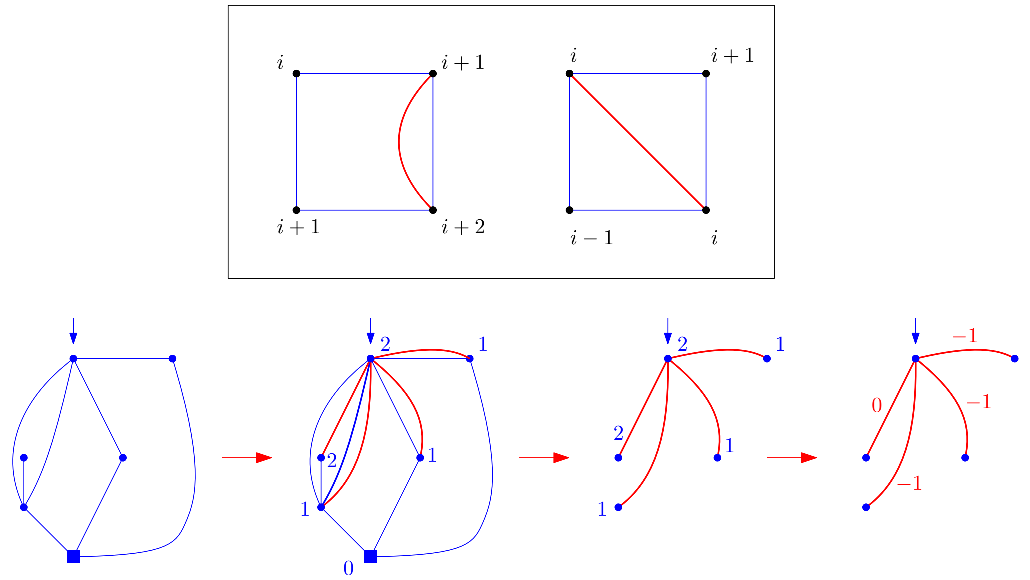


FIGURE 1.11 – La bijection de Cori–Vauquelin–Schaeffer, sur un exemple. Partant d'une quadrangulation planaire, on étiquette chaque sommet par sa distance au sommet marqué. On trace ensuite des arêtes dans chaque face suivant les règles dans l'encadré. On oublie alors les arêtes de la quadrangulation de départ. Le résultat est un arbre étiqueté sur ses sommets, qu'on peut réétiqueter sur ses arêtes par l'incrément de l'étiquette dans la direction de la racine. Pour chaque arbre, il y a deux manières de revenir en arrière.

les cartes planaires avec un modèle d'arbres étiquetés (voir Figure 1.11). Plus précisément, soit M_n l'ensemble des cartes planaires à n arêtes, soit $Q_n^\bullet$ l'ensemble des quadrangulations planaires à n faces avec un sommet distingué, et soit $\mathcal{T}_n$ l'ensemble des arbres planaires à n arêtes dont les arêtes sont étiquetées par 0, 1, ou -1. La bijection de Cori–Vauquelin–Schaeffer fournit une correspondance entre $Q_n^\bullet$ et $2 \times \mathcal{T}_n$, et cela permet entre autres de redémontrer la formule de Tutte, puisqu'une quadrangulation planaire à n faces possède $n + 2$ sommets, et que la proposition 1.1.5 fournit une bijection entre cartes planaires et quadrangulations planaires. Cette bijection a été généralisée et adaptée à d'autres modèles de cartes [BDFG04], et adaptée au genre supérieur [CMS09], ainsi qu'aux cartes avec plusieurs sommets marqués [Mie09].

Une autre famille de bijections, entre des cartes et des arbres bourgeonnants a été initiée par Schaeffer [Sch98], et là encore, généralisée [AP15, Lep19]. D'autre part, les cartes unicellulaires[7] de genre supérieur sont en bi-

[7]c-à-d n'ayant qu'une seule face.

jection avec des arbres décorés par des permutations [CFF13]. Enfin, il existe une bijection pour les cartes munies d'un arbre couvrant [Ber07], qui a été généralisée en genre supérieur [BC11] ainsi qu'aux cartes à degrés et maille prescrits [BF12].

Bijections arboricoles mises à part, des bijections ont été découvertes entre des cartes et d'autres objets combinatoires classiques[8] comme les intervalles de Tamari [BB09] ou permutations de Baxter [BBMF11].

Enfin, pour finir ce petit tour d'horizon des bijections sur les cartes, il convient de mentionner les bijections entre cartes elles-mêmes : que ce soit entre différents modèles de cartes [ArB13] ou pour démontrer des formules de récurrence [Bet14].

Pour les combinatoristes, ces liens bijectifs placent les cartes au centre d'une combinatoire riche, ce qui en fait des objets fascinants et pertinents. Du point de vue probabiliste, les bijections sont à la base de nombreuses approches des cartes aléatoires, comme on le verra dans la section 1.2.4.

1.2.3 L'approche par caractères et la hiérarchie KP

Comme mentionné précédemment, les cartes ne sont pas seulement des objets géométriques : elles peuvent être interprétées comme des factorisations de permutations, et il s'avère qu'en théorie des représentations il existe justement une formule « universelle » qui permet de compter les factorisations dans un groupe en fonction de ses caractères — et ce de manière assez fine. On donnera un exposé « autocontenu » de cette formule (appelée suivant les auteurs « formule de Burnside » ou « formule de Frobenius ») dans le chapitre 2.

Dans certains cas, cette formule se simplifie suffisamment pour donner des formules explicites : le premier exemple connu est une formule pour certains nombres de Hurwitz [Jac88], qui comptent des cartes étiquetées à une face, ou, de manière équivalente, certains revêtements ramifiés de la sphère à homéomorphisme près. On peut également citer la récurrence des quadrangulations [JV90], généralisée par la suite aux constellations [Fan14] ainsi que la formule de Goupil–Schaeffer comptant les cartes biparties à une face à degrés prescrits [GS98] (pour les constellations, voir [PS02]).

L'approche par caractères ne donne pas toujours de résultats explicites directement, mais elle permet de montrer un résultat très général : la série génératrice des cartes vérifie les *hiérarchies KP et 2-Toda*. Ce résultat a été démontré notamment par Goulden et Jackson en 2008 [GJ08]. Il était « connu » en physique depuis les années 1990 par diverses méthodes au dé-

[8] on ne prétend pas ici en faire une liste exhaustive.

FIGURE 1.12 – Le phare des Baleines à l'Ile de Ré. ©Michel Griffon

part non rigoureuses (voir [LZ04] et les références qui y sont donnéees). Les méthodes d'Okounkov sur les objets proches que sont les nombres de Hurwitz [Oko00] conduisent également à ce résultat.

Il s'agit de familles d'équations qui ont été étudiées en premier lieu en physique mathématique. A l'origine, l'équation de Kadomtsev–Petviashvili [KP70] est une équation aux dérivées partielles décrivant des phénomènes non linéaires de mouvement des vagues, comme par exemple l'apparition de vagues « orthogonales » que l'on peut voir sur la figure 1.12.

Afin d'étudier les symétries de cette équation, on peut construire une famille infinie d'équations en une infinité de variables $(p_1, p_2, \ldots)$: la hiérarchie KP. La première équation de la hiérarchie KP (qui correspond à l'équation originelle) s'écrit

$$F_{3,1} = F_{2,2} + \frac{1}{2}F_{1,1}^2 + \frac{1}{12}F_{1,1,1,1},$$

où les indices correspondent à des dérivées partielles, par exemple on a $F_{3,1} = \frac{\partial^2}{\partial p_1 \partial p_3}F$. Cette hiérarchie admet une généralisation incluant une deuxième suite infinie de variables : la hiérarchie 2-Toda.

Ce sont des hiérarchies intégrables, ce qui signifie qu'on peut décrire explicitement l'espace des solutions. Elles possèdent des propriétés algébriques fortes et s'appliquent à un grand nombre de modèles, dont les cartes. L'avantage de ces hiérarchies est qu'elles impliquent des équations aux dérivées partielles sur les séries génératrices, ce qui permet par la suite de dériver des formules de récurrence sur les cartes [GJ08, CC15, KZ15]. On trouvera une introduction à ce sujet dans le chapitre 2 et une application dans le chapitre 5. On termine cette section en donnant la formule de Goulden–Jackson [GJ08] qui est la première formule de récurrence « issue de KP ». Soit $\tau(n,g)$ le nombre de triangulations de genre g à $2n$ faces, alors

$$\begin{aligned}(n+1)\tau(n,g) =& 4n(3n-2)(3n-4)\tau(n-2,g-1) + 4(3n-1)\tau(n-1,g) \\ &+ 4 \sum_{\substack{i+j=n-2 \\ i,j \geqslant 0}} \sum_{\substack{g_1+g_2=g \\ g_1,g_2 \geqslant 0}} (3i+2)(3j+2)\tau(i,g_1)\tau(j,g_2) + 2\mathbb{1}_{n=g=1}.\end{aligned} \tag{1.2.2}$$

Cette formule permet de calculer les nombres $\tau(n,g)$ beaucoup plus efficacement que toute autre méthode connue.

1.2.4 L'étude asymptotique des cartes aléatoires

Comme les graphes, les cartes, en tant structures discrètes, se prêtent bien à une étude probabiliste et asymptotique. La question fondamentale est « à quoi ressemble une grande carte prise au hasard ? ». Le principe sera le suivant : on se fixe un modèle de cartes, une distribution de probabilité sur ces cartes, et on cherche à connaître le comportement asymptotique d'une certaine observable ou bien une limite probabiliste (selon une topologie donnée) quand la taille des cartes tend vers l'infini. La plupart des travaux sur les cartes aléatoires (incluant ceux dont on va parler dans les chapitres suivants) établissent des résultats *en loi*[9]. On rappelle qu'une suite $(X_n)_{n \geqslant 0}$ à valeurs dans un espace métrique $\mathcal{X}$ converge en loi vers une variable X si pour toute fonction ϕ continue bornée à valeurs réelles, on a :

$$\mathbb{E}(\phi(X_n)) \xrightarrow[n\to\infty]{} \mathbb{E}(\phi(X)).$$

Cette définition dépend de la notion de continuité, et donc du choix d'une topologie sur l'espace $\mathcal{X}$. L'un des premiers résultats significatifs dans l'étude asymptotique des cartes aléatoires est la mesure du diamètre par Chassaing et Schaeffer : ils établissent qu'une carte à n arêtes tirée uniformément aura, avec probabilité tendant vers 1, un diamètre d'ordre $n^{1/4}$ quand $n \to \infty$[10]. Ce résultat fut le premier d'un ensemble de travaux de divers auteurs s'étendant sur une décennie et aboutissant à la convergence d'échelle des cartes planaires uniformes vers la *carte brownienne*. Il s'agit sûrement du résultat le plus célèbre dans le domaine des cartes aléatoires, et il a été démontré en

[9] Il existe également des articles qui établissent des convergences plus fortes sur les cartes, voir par exemple [AB14].

[10] l'article montre même la convergence du profil.

2011 indépendamment par Le Gall et Miermont [LG13, Mie13]. Plus précisément, les cartes munies de la distance de graphe sont des espaces métriques qu'on munit de la mesure uniforme sur leur sommets. D'après le résultat de Chassaing–Schaeffer, si on « met à l'échelle » la distance de graphe en la divisant par $n^{1/4}$, une carte typique aura un diamètre borné, ce qui en fait un espace métrique compact. En considérant les cartes uniformes de taille[11] n pour tout n on obtient une suite d'espaces métriques aléatoires compacts. Ces objets ont une limite (en loi) dite de Gromov–Hausdorff–(Prokoroff) qui est la carte Brownienne. C'est un objet fractal (de dimension de Hausdorff 4 [LG07]) qui est néanmoins homéomorphe à la sphère de dimension 2 [LGP08]. Les travaux sur la convergence d'échelle des cartes sont une parfaite illustration de la notion d'*universalité* (qui nous intéressera dans le chapitre 7) : quel que soit le modèle de cartes, on observe des phénomènes similaires (asymptotiquement). En l'occurrence, il a été démontré que la plupart des modèles de cartes planaires « raisonnables » (triangulations, quadrangulations, ...) convergent vers la carte Brownienne[12] [BJM14, ABA17, Mar18]. Des travaux similaires à genre quelconque fixé existent également [Bet10, Bet12].

Il existe une autre façon d'étudier les limites de grandes cartes : la *limite locale*, c'est-à-dire la convergence par rapport à la distance locale d_{loc} que l'on a définie en Section 1.1.5. Schématiquement, il s'agit de décrire la loi limite du voisinage de la racine. Cette notion est très proche de la limite de Benjamini–Schramm pour les graphes, introduite dans [BS01].

La question est la même que pour la limite d'échelle : étant donné une suite de cartes aléatoires uniformes, y a-t-il convergence (locale), et peut-on décrire l'objet limite ? Le premier travail du genre date de 2002 : Angel et Schramm [AS03] ont étudié les triangulations planaires aléatoires uniformes, et ont démontré qu'il y avait convergence locale vers une triangulation infinie du plan qu'ils ont nommé l'UIPT (voir Figure 1.13), pour Uniform Infinite Planar Triangulation (qui sera définie dans le Chapitre 3). Par la suite, plusieurs recherches ont été consacrées à l'UIPT en elle même, et de nombreuses propriétés ont été étudiées, par exemple la croissance et la percolation [Ang03] ou la récurrence de la marche aléatoire [GGN13].

Pour la convergence locale, il n'existe pas d'objet limite universel (comme la carte brownienne l'est pour les limites d'échelle de cartes). En effet, une suite de quadrangulations ne peut pas converger localement vers une triangulation infinie. En revanche, on peut toujours parler d'universalité puisque, pour de nombreux modèles, on a une convergence locale vers des cartes infinies du plan qui ont de nombreuses propriétés en commun. Les quadrangula-

[11]la taille est définie différemment selon les modèles : le nombre d'arêtes, de faces, ...
[12]à une constante près.

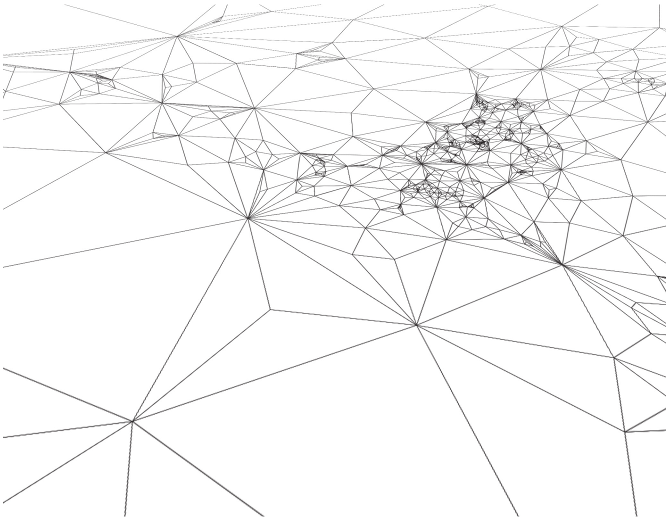


FIGURE 1.13 – Une vue de dessus de l'UIPT, par Igor Kortchemski.

tions ont été étudiées (et sans suprise, l'objet limite s'appelle l'UIPQ), ainsi que quelques autres modèles, et Budd a fourni une étude assez exhaustive des limites de cartes biparties de Boltzmann [Bud15] (la convergence locale a été démontrée dans [Ste18]). Par la suite, on a découvert que l'objet limite, l'IBPM[13], a des propriétés très similaires à l'UIPT [BC17], ce qui constitue un résultat d'universalité satisfaisant car une vaste famille de cartes a été couverte.

La question du genre supérieur se pose naturellement. Pour $g > 0$ fixé, les triangulations uniformes de genre g et de taille n convergent vers l'UIPT quand $n \to \infty$: on ne « voit » pas le genre localement (c'est un fait qui était bien connu, bien qu'écrit nulle part). Quand le genre tend vers l'infini, la question de la convergence a fait l'objet d'une conjecture de Benjamini et Curien à laquelle nous répondons dans le chapitre 6 pour les triangulations, et la question de l'universalité est couverte dans le chapitre 7.

1.2.5 D'autres approches des cartes

Jusqu'ici cette présentation des travaux antérieurs sur les cartes s'est limitée aux domaines qui sont directement liés aux travaux présentés dans cette thèse. Afin de rendre justice à la fois à la puissance des cartes ainsi qu'aux autres domaines de recherche dans lesquelles elle interviennent, on va présenter rapidement d'autres approches des cartes[14].

[13]pour Infinite Boltzmann Planar Map.
[14]sans avoir la prétention d'en faire un traitement exhaustif.

Dans le domaine de l'informatique, les cartes sont très utilisées en *géométrie algorithmique* : de nombreux problèmes algorithmiques sur les graphes possèdent des solutions plus efficaces propres aux graphes plongeables sur une surface donnée (c'est-à-dire quand on borne le genre de la carte correspondante), à l'inverse, étant donné un graphe on peut vouloir savoir quel est le genre minimal de la surface sur laquelle on peut le plonger, ou bien « simplifier sa topologie » (le modifier un peu pour le rendre planaire par exemple). Les cartes interviennent également en *informatique graphique* (dans l'étude des maillages 3D par exemple), ou bien en *analyse topologique des données*. Le chapitre 3 de [CdV12] fait un tour d'horizon très complet de l'usage des cartes dans ces domaines.

Les cartes sont également omniprésentes en physique mathématique. Un exemple célèbre est la preuve de la *conjecture de Witten* [Wit91] par Kontsevich [Kon92]. La conjecture porte sur l'espace de modules des courbes complexes, il s'agit de montrer qu'une certaine fonction de partition (c'est-à-dire série génératrice) des intersections sur cet espace vérifie la hiérarchie KdV (dont la hiérarchie KP est une généralisation !). La preuve utilise une correspondance entre cet espace et l'espace de modules des *graphes rubans* (c'est-à-dire des cartes). Cette preuve, ainsi que les techniques développées pour y parvenir, ont ouvert la voie à un grand nombre de travaux en *théorie de l'intersection* et en *géométrie énumérative*. On en trouvera un exemple récent à la fin de cette section. La conjecture de Witten est motivée par des considérations de *gravité quantique* (ou plutôt des tentatives d'en établir une théorie robuste en deux dimensions). On retrouve les cartes dans une autre approche de la gravité quantique : il a été démontré que la carte brownienne était en quelque sorte équivalente à la gravité quantique de Liouville en 2D pour le paramètre $\gamma = \sqrt{8/3}$[15], et les recherches dans ce domaine sont toujours très actives (aussi bien à partir des cartes que directement dans le continu). Pour une bonne introduction à ces questions, on pourra lire [Mil18, Gwy19].

Enfin, dans le domaine des mathématiques, les cartes se retrouvent également en géométrie algébrique, avec les *dessins d'enfant* de Grothendieck [Gro97] qui ne sont rien d'autre que des cartes biparties. Il existe une correspondance entre les dessins d'enfant et certaines courbes algébriques définies sur $\overline{\mathbb{Q}}$, qui permet d'étudier de manière combinatoire l'action du groupe de Galois absolu des rationnels sur ces courbes (pour une introduction à ces objets, voir [LZ04], chapitre 2). On peut également citer l'étude des *billards polygonaux rationnels*, qui nécessite en particulier de calculer des volumes d'espaces de modules des *surfaces de translations* (voir par exemple [Zor06]). Ce calcul est possible via l'énumération de *surfaces à petits carreaux* qui

[15] plus précisément, la carte détermine la structure conforme et réciproquement.

jouent le rôle de points entiers dans ces espaces (voir par exemple [Mat18]). Une méthode d'énumération consiste à compter les métriques entières sur des cartes, grâce aux méthodes développées par Kontsevich [AEZ14].

1.3 Contributions et organisation du manuscrit

Le travail de ma thèse est à l'intersection des différents domaines de l'étude des cartes présentés dans l'introduction : énumération, approche bijective, méthodes algébriques (et hiérarchies KP/2-Toda) et étude asymptotique des cartes aléatoires. A plusieurs occasions il s'agira de combiner ces différentes approches pour démontrer les résultats voulus.

De plus, comme on peut le voir dans l'historique, une majorité des travaux sur les cartes se focalisent sur les cartes planaires, tandis que mon travail porte principalement sur les cartes « en tout genre »[16], voire même dont le genre tend vers l'infini.

A partir du chapitre suivant, cette thèse sera rédigée en anglais. La suite du manuscrit s'articule ainsi :

- Chapitre 2 : introduction aux hiérarchies KP et 2-Toda,
- Chapitre 3 : introduction aux limites locales et cartes infinies du plan,
- Chapitre 4 : preuves bijectives de formules de récurrences issues de la hiérarchie KP,
- Chapitre 5 : formules de récurrence pour les cartes biparties à degrés prescrits et constellations,
- Chapitre 6 : limites locales de triangulations de grand genre,
- Chapitre 7 : limites locales de cartes biparties à degrés prescrits en grand genre,
- Chapitre 8 : quelques problèmes ouverts.

Les chapitres 4 à 7 sont chacun adaptés d'un article (la structure de chaque article reste globalement la même, seules l'introduction et les définitions sont modifiées et allégées). Pour terminer cette introduction, on fait un bref résumé du contenu de chacun de ces chapitres.

[16] même dans le Chapitre 4, dont l'objet principal est les cartes planaires, la motivation principale est la démonstration bijective de la formule de Goulden–Jackson (1.2.2) qui est valable en tout genre, on trouvera d'ailleurs une ébauche de bijection pour des cartes en genre supérieur.

1.3.1 Chapitre 4 : des bijections

Ce chapitre est basé sur l'article *A new family of bijections for planar maps*, paru dans *Journal of Combinatorial Theory, Series A* [Lou19a].

Ce travail s'inscrit dans un objectif plus large de fournir une bijection unifiée expliquant les formules de récurrence issues de la hiérarchie KP, comme par exemple les formules de Goulden–Jackson (1.2.2) et Carrell–Chapuy [CC15]. Ces formules permettent d'énumérer les cartes en tout genre. On donne ici des bijections pour le cas planaire, qui capture la nature quadratique des formules générales.

Soit $Q(n, f)$ le nombre de cartes planaires à n arêtes et f faces. On démontre le théorème suivant :

Théorème 1.3.1. *Il existe deux bijections, basées sur une opération dite de* cut-and-slide*, qui consiste à couper des arêtes puis les décaler selon un chemin « en profondeur » dans la carte, qui démontrent les formules suivantes :*

$$(f-1)Q(n,f) = \sum_{\substack{i+j=n-1\\ i,j\geqslant 0}} \sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}} v_1 Q(i,f_1)(2j+1)Q(j,f_2), \tag{1.3.1}$$

où $v_1 = 2 + i - f_1$ (v_1 compte des sommets d'après la formule d'Euler), et

$$vQ(n,f) = 2(2n-1)Q(n-1,f) + \sum_{\substack{i+j=n-1\\ i,j\geqslant 0}} \sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}} v_1 Q(i,f_1) v_2 Q(j,f_2), \tag{1.3.2}$$

où $v = 2 + n - f$, $v_1 = 2 + i - f_1$ et $v_2 = 2 + j - f_2$.

Ces deux formules impliquent la formule de Carrell–Chapuy dans le cas planaire, qui s'écrit :

$$\begin{aligned}(n+1)Q(n,f) =& 2(2n-1)Q(n-1,f) + 2(2n-1)Q(n-1,f-1)\\ &+ 3 \sum_{\substack{i+j=n-2\\ i,j\geqslant 0}} \sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}} (2i+1)(2j+1)Q(i,f_1)Q(j,f_2). \end{aligned}\tag{1.3.3}$$

La bijection impliquant (1.3.2) est une généralisation de la bijection de Rémy sur les arbres planaires [Rém85] à toutes les cartes planaires.

La bijection étant très explicite, et ne modifiant que peu les degrés des sommets, on trouve également une version de (1.3.1) qui permet de compter les cartes planaires quand on fixe le nombre de sommets de chaque degré. La formule de Goulden–Jackson (1.2.2) en est une conséquence directe.

Un autre cas particulier a été traité auparavant, celui des cartes à une face [CFF13], qui correspond à la célèbre formule de Harer–Zagier [HZ86].

On notera que les formules dans ce cas ne sont plus quadratiques mais deviennent linéaires. Le cas général, les cartes de genre quelconque avec un nombre quelconque de faces reste largement ouvert, cependant on fournit également une ébauche de bijection pour un cas très particulier : les cartes dites « précubiques » (sommets de degrés 1 ou 3) à deux faces, de genre quelconque.

1.3.2 Chapitre 5 : des formules

Ce chapitre est basé sur l'article *Simple formulas for constellations and bipartite maps with prescribed degrees* à paraître dans *Canadian Journal of Mathematics* [Lou19b].

Dans la lignée des formules de Goulden–Jackson et Carrell–Chapuy, on démontre une formule de récurrence pour un modèle de cartes assez vaste : les cartes biparties à degré prescrits :

Théorème 1.3.2. *Soit $\beta_g(\mathbf{f})$ le nombre de cartes biparties de genre g avec f_i faces de degré $2i$ (pour $\mathbf{f} = (f_1, f_2, \ldots)$). Alors, on a*

$$\binom{n+1}{2}\beta_g(\mathbf{f}) = \sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ g_1+g_2+g^*=g}} (1+n_1)\binom{v_2}{2g^*+2}\beta_{g_1}(\mathbf{s})\beta_{g_2}(\mathbf{t}) + \sum_{g^*\geqslant 0}\binom{v+2g^*}{2g^*+2}\beta_{g-g^*}(\mathbf{f}) \tag{1.3.4}$$

où $n = \sum_i i f_i$, $n_1 = \sum_i i s_i$, $v = 2-2g+n-\sum_i f_i$, $v_2 = 2-2g_2+n_2-\sum_i t_i$ et $n_2 = \sum_i i t_i$ (les « n's » comptent des arêtes, les « v's » comptent des sommets, en accord avec la formule d'Euler), en adoptant la convention $\beta_g(\mathbf{0}) = 0$.

Cette formule est, comme les précédentes formules pour les cartes, quadratique, et permet de calculer tous les nombres de cartes biparties à degré prescrits quel que soit le genre, c'est même la manière la plus rapide de le faire que l'on connaisse à ce jour. On démontre également des formules similaires pour les constellations et les nombres de Hurwitz monotones (voir Chapitre 5 pour une définition).

La formule (1.3.4) est un outil crucial dans l'étude des limites locales des cartes de grand genre faite en Chapitre 7.

La démonstration diffère un peu des formules précédentes pour les cartes [GJ08, CC15, KZ15] car elle s'appuie sur la hiérarchie 2-Toda, et non la hiérarchie KP (la première étant une généralisation de la seconde). Il s'agit d'une preuve assez technique sur une série génératrice bien choisie. Elle commence par des manipulations algébriques inspirées par les travaux d'Okounkov [Oko00] et Dubrovin–Yang–Zagier [DYZ17] sur les nombres de Hurwitz, puis des considérations combinatoires propres au modèle des cartes biparties.

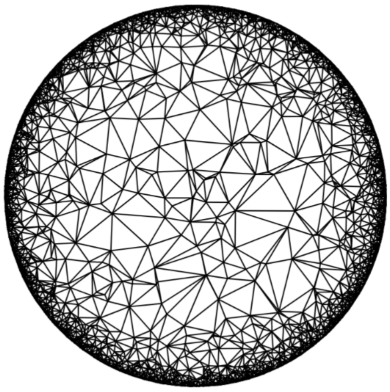


FIGURE 1.14 – Une simulation de la PSHT plongée dans le plan hyperbolique, par Nicolas Curien.

1.3.3 Chapitre 6 : des triangulations de grand genre

Ce chapitre est basé sur l'article *Local limits of high genus triangulations* avec Thomas Budzinski (soumis) [BL19].

Il s'agit d'une preuve de la conjecture de Benjamini et Curien [Cur16] qui concerne la limite locale des triangulations de grand genre. Curien a introduit une famille de triangulations infinies du plan à un paramètre λ, les PSHT $\mathbb{T}_\lambda$ (voir Figure 1.14, et Chapitre 3 pour une définition). Pour tout $\lambda \in (0, \lambda_c]$[17], soit $h \in \left(0, \frac{1}{4}\right]$ tel que $\lambda = \frac{h}{(1+8h)^{3/2}}$, on définit

$$d(\lambda) = \frac{h \log \frac{1+\sqrt{1-4h}}{1-\sqrt{1-4h}}}{(1+8h)\sqrt{1-4h}}.$$

Soit (g_n) une suite d'entiers telle que $\frac{g_n}{n} \to \theta$, avec $\theta \in [0, \frac{1}{2}[$. Soit T_n une triangulation uniforme de genre g_n à $2n$ faces, alors :

Théorème 1.3.3.

$$T_n \to \mathbb{T}_\lambda$$

en loi pour la topologie locale, où λ est l'unique solution de

$$d(\lambda) = \frac{1-2\theta}{6}.$$

L'idée de la conjecture est la suivante : dans une triangulation dont le genre est linéaire en la taille, le degré moyen d'un sommet est asymptotiquement strictement supérieur à 6, qui est la valeur moyenne dans le cas des

[17] où $\lambda_c = \frac{1}{12\sqrt{3}}$ est le rayon de convergence de la série des triangulations planaires

triangulations planaires. D'un autre côté, Curien a introduit les PSHT, une famille de triangulations infinies du plan qui sont en quelque sorte des « déformations hyperboliques » de l'UIPT d'Angel et Schramm (ce qui implique, entre autres, que le « degré moyen » y est là aussi strictement supérieur à 6). Il est alors naturel de se demander si l'un est la limite de l'autre.

La preuve est très différente de celle d'Angel et Schramm [AS03] pour les triangulations planaires, qui repose fortement sur l'existence de résultats d'asymptotique précis sur l'énumération. Dans le cas des cartes de grand genre, il nous faudrait de l'asymptotique sur une suite bivariée dont les deux paramètres tendent vers l'infini en même temps, et il n'existe pas (encore) d'approche directe. Le seul outil d'énumération à notre disposition est la formule de Goulden–Jackson (1.2.2).

La première partie de la preuve consiste en un résultat de *tension*, c'est-à-dire que toute sous-suite de T_n possède une sous-sous-suite convergente. Par des arguments classiques, la preuve est équivalente à montrer que le degré de la racine est presque sûrement fini asymptotiquement, ce qui est fait grâce à un argument combinatoire de chirurgie locale, le *Bounded Ratio Lemma*. On démontre également, grâce à la formule de Goulden–Jackson (1.2.2), que toute limite potentielle est forcément planaire et « one-ended » (voir Chapitre 3 pour une définition).

Dans la deuxième partie, on démontre que les limites potentielles sont d'une forme très particulière : ce sont des mélanges de PSHT, c'est-à-dire qu'il existe une variable aléatoire réelle Λ telle que la limite a la loi de $\mathbb{T}_\Lambda$ (c'est donc une carte aléatoire dont le paramètre lui même est aléatoire).

La dernière partie de la preuve consiste à démontrer que ce fameux paramètre Λ est en fait déterministe et dépend uniquement de θ, le ratio asymptotique entre le genre et la taille. Pour ce faire, on démontre par un argument mélangeant combinatoire et probabilités, *l'argument des deux trous*, que « l'hyperbolicité est répartie uniformément dans la carte » , puis on termine en montrant qu'on peut « lire » le paramètre Λ directement sur la PSHT grâce à l'inverse du degré de la racine.

En guise de corollaire, la démonstration de la conjecture nous offre des estimées asymptotiques sur le nombre de cartes de grand genre. On rappelle que $\tau(n, g)$ compte le nombre de triangulations de genre g à $2n$ triangles, alors :

Théorème 1.3.4. *Soit (g_n) une suite telle que $0 \leqslant g_n \leqslant \frac{n+1}{2}$ pour tout n et $\frac{g_n}{n} \to \theta \in \left[0, \frac{1}{2}\right]$. Alors*

$$\tau(n, g_n) = n^{2g_n} \exp\left(f(\theta)n + o(n)\right)$$

quand $n \to +\infty$*, avec* $f(0) = \log 12\sqrt{3}$*, et* $f(1/2) = \log \frac{6}{e}$ *et*

$$f(\theta) = 2\theta \log \frac{12\theta}{e} + \theta \int_{2\theta}^{1} \log \frac{1}{\lambda(\theta/t)} \mathrm{d}t$$

pour $0 < \theta < \frac{1}{2}$.

1.3.4 Chapitre 7 : de l'universalité

Ce chapitre est basé sur l'article *Universality for local limits of high genus maps* avec Thomas Budzinski (en cours d'écriture) [BL20].

On y généralise le travail précédent : une fois la conjecture de Benjamini et Curien démontrée, la question de l'universalité se pose naturellement, et les objets naturels sont les cartes biparties à degrés prescrits, dont la convergence locale a déjà été étudiée dans le cas planaire [Ste18, Bud15]. Dans le cas général, le potentiel objet limite est l'*Infinite Boltzmann Planar Map* (IBPM) $\mathbb{M}_{\mathbf{q}}$, qui est une carte infinie du plan à une infinité de paramètres $\mathbf{q} = (q_1, q_2, \ldots)$ (voir Chapitre 3 pour une définition). Selon certaines conditions sur $\mathbf{q}$, on se trouve soit dans le cas dit *critique* — $\mathbb{M}_{\mathbf{q}}$ correspond alors à une limite locale de cartes planaires — ou dans le cas dit *sous-critique*, et $\mathbb{M}_{\mathbf{q}}$ possède des propriétés hyperboliques.

Il restait alors à démontrer le théorème suivant :

Théorème 1.3.5. *Soit* (g_n) *une suite vérifiant* $\frac{g_n}{n} \to \theta$ *avec* $\theta \in \left[0, \frac{1}{2}\right)$*, et* $\mathbf{f}^{(n)} = (f_1^{(n)}, f_2^{(n)}, \ldots)$ *telle que* $\sum_{i \geqslant 1} i f_i^{(n)} = n$*. On suppose que, pour tout* i*, il existe* α_i *tel que* $\frac{f_i^{(n)}}{n} \to \alpha_i$*, et qu'on a* $\sum_i i\alpha_i = 1$ *et* $\sum_i i^2 \alpha_i < \infty$*.*

Soit $\mathbf{M}_n$ *uniforme parmi les cartes biparties de genre* g_n *dont les degrés des faces sont donnés par* $\mathbf{f}^{(n)}$*. Alors*

$$\mathbf{M}_n \xrightarrow[n \to +\infty]{(d)} \mathbb{M}_{\mathbf{q}}$$

pour la topologie locale, où la suite $\mathbf{q}$ *est une fonction déterministe et injective de* θ *et des* α_i*.*

La preuve suit le même schéma global que pour les triangulations, cependant les preuves sont beaucoup plus techniques que dans le chapitre précédent. La difficulté par rapport aux triangulations tient à deux raisons principales : on doit désormais gérer un nombre infini de paramètres au lieu d'un seul, et les faces sont de taille non bornée. Chaque argument demande de nouvelles idées pour être adapté depuis les triangulations. La fin de la preuve est même totalement différente puisqu'il devient impossible de calculer l'inverse du degré de la racine. Cependant, le fait que la structure globale de la preuve soit la même montre la robustesse de l'approche développée dans le travail précédent.

Une fois encore, on obtient des résultats d'énumération asymptotique :

Théorème 1.3.6. *Soit* (g_n) *et* $\mathbf{f}^{(n)}$ *définis comme dans le théorème 1.3.5. Soit* $\beta_g(\mathbf{f})$ *le nombre de cartes biparties de genre* g *dont les degrés des faces sont donnés par* $\mathbf{f}$*. Alors*

$$\beta_{g_n}(\mathbf{f}^{(n)}) = n^{2g_n} \exp\left(f(\theta, (\alpha_i)_{i\in\mathbb{N}})n + o(n)\right)$$

où f *est une fonction explicite.*

Il faut également noter que ce résultat repose sur le théorème 1.3.2 (là où on avait utilisé la formule de Goulden–Jackson (1.2.2) pour démontrer le théorème 1.3.3). Ce chapitre illustre bien l'apport mutuel entre combinatoire et probabilités. D'un côté, une formule énumérative, obtenue par des techniques de combinatoire algébrique, sert à démontrer un résultat sur des cartes aléatoires. De l'autre, un travail probabiliste a pour corollaire un résultat d'énumération asymptotique des cartes.

Chapter 2

The KP and 2-Toda hierarchies

2.1 A quick introduction to representation theory and symmetric functions

Here we introduce some results about representation theory of the symmetric group $\mathfrak{S}_n$ and symmetric functions. We refer to [Sta99] for a clear exposition of all these results. For results of representation theory of finite groups, we refer to [Ser77].

2.1.1 Representation theory

Basics of representation theory. Let G be a finite group. A (complex) representation of G is a complex, finite dimensional vector space V_ρ along with a linear morphism $\rho : G \to GL(V_\rho)$. A representation V_ρ is said to be irreducible if there does not exist two representations $V_{\rho'}$ and $V_{\rho''}$ (of dimensions > 0) such that $V_\rho = V_{\rho'} \oplus V_{\rho''}$. The group algebra $\mathbb{C}[G]$ is a natural representation of G (G acts on it by multiplication on the left). It decomposes this way:

$$\mathbb{C}[G] = End(V_\rho),$$

where the sum spans over all irreducible representations ρ of G. The *character* of a representation ρ applied to an element $g \in G$ is

$$\chi^\rho(g) = tr_{V_\rho}(\rho(g)).$$

We have $\chi^\rho(1) = \dim V_\rho$. If g and g' are in the same conjugacy class, then $\chi^\rho(g) = \chi^\rho(g')$. Characters can be extended by linearity to all elements of $\mathbb{C}[G]$, and if $x \in \mathbb{C}[G]$ is *central*, that is, it commutes with all $y \in \mathbb{C}[G]$, then

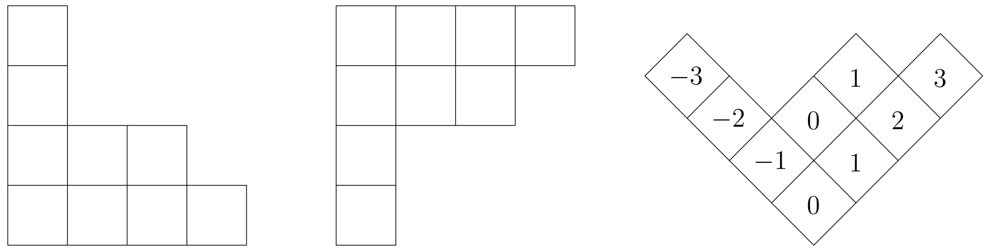


Figure 2.1 – The Ferrers diagram of the partition $(4, 3, 1, 1)$ in French, English and Russian convention. In the third diagram, the content of each box is given.

for all $y \in \mathbb{C}[G]$:

$$\chi^\rho(xy) = \frac{\chi^\rho(x)}{\dim V_\rho} \cdot \chi^\rho(y). \tag{2.1.1}$$

Representations of $\mathfrak{S}_n$. From now on, we will only consider $G = \mathfrak{S}_n$. First, we need to define integer partitions.

Definition 2.1.1. A *partition* is a finite nonincreasing sequence of strictly positive integers. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is a partition, then we say that the λ_i's are the *parts* of λ, we say that the *length* of λ is $l(\lambda) := \ell$ (the number of parts), and $n = \lambda_1 + \lambda_2 + \ldots + \lambda_\ell$ is the *size* of λ. We also say that λ is a partition of n, and we write $|\lambda| = n$ or $\lambda \vdash n$. The *empty partition* is written $\emptyset$.

The *Ferrers diagram* (see Figure 2.1) of a partition λ is made of rows of boxes, the k-th row having λ_k boxes, such that the first boxes of the rows are aligned on the left. In the French convention, the rows go from bottom to top, whereas in the American convention, they go from top to bottom. There exists a third way of drawing these diagrams, the Russian convention, that is the French convention, rotated by 45 degrees counter-clockwise.

The *content* $c(\square)$ of a box $\square \in \lambda$ is the abscissa of the the center of $\square$ in the Ferrers diagram of λ in Russian convention (see Figure 2.1 right).

The conjugacy classes of $\mathfrak{S}_n$ are indexed by partitions: C_μ is the class of permutations that have cycle-type μ. Since the characters are constant inside conjugacy classes, for $\sigma \in \mathbb{C}_\mu$, and a character χ, we write $\chi(\mu) = \chi(\sigma)$.

Proposition 2.1.1 (Classical, see [Sta99])**.** *The irreducible representations of $\mathfrak{S}_n$ are indexed by partitions of n. We will write V_λ the representation associated to λ, χ^λ is the character of V_λ, and* $\dim \lambda = \dim(V_\lambda)$.

$$\begin{array}{|c|c|c|c|}
\cline{1-1} 3 \\
\cline{1-1} 1 \\
\cline{1-3} 1 & 2 & 2 \\
\hline 1 & 1 & 2 & 2 \\
\hline
\end{array}$$

Figure 2.2 – A border strip tableau of shape $(4, 3, 1, 1)$ and type $(4, 4, 1)$.

There is a combinatorial way to calculate the characters of $\mathfrak{S}_n$: the *Murnaghan–Nakayama rule*. We first need some definitions.

Definition 2.1.2. A *skew partition* $\mu \setminus \lambda$ is defined by two partitions μ and λ such that $\mu_i \geqslant \lambda_i$ for all i. The diagram associated to $\mu \setminus \lambda$ is constructed by removing the diagram of λ to the diagram of μ. A skew partition is *connected* if the interior of its diagram (seen as a union of solid squares) is a connected open set. A *border strip* is a connected skew partition whose diagram does not contain a 2 by 2 square. The *height* $ht(S)$ of a border strip S is its number of rows minus one.

A *border strip tableau* (see Figure 2.2) T of shape λ and type μ is a diagram of shape λ filled with integers from 1 to $l(\mu)$, such that the rows and columns are increasing and that the number i appears exactly μ_i times. Moreover, for all i, the diagram T_i consisting of all boxes with number i must be a border strip. Finally, we set $ht(T) = ht(T_1) + ht(T_2) + \ldots + ht(T_{l(\mu)})$.

Theorem 2.1.2 (Murnaghan–Nakayama)**.** *The irreducible characters of* $\mathfrak{S}_n$ *can be calculated by the following formula:*

$$\chi^\lambda(\mu) = \sum_T (-1)^{ht(T)},$$

where the sum spans over all border strip tableaux T of shape λ and type μ.

Now we can present the Frobenius formula, that counts factorizations of permutations, but first we need to introduce the *Jucys–Murphy elements*.

Definition 2.1.3. The Jucys–Murphy elements are elements of $\mathbb{C}[\mathfrak{S}_n]$ defined in the following way:

$$J_i = \sum_{k<i} (k, i) \quad \forall 1 \leqslant i \leqslant n.$$

Proposition 2.1.3 ([Mur81])**.** *Let f be a symmetric polynomial in n variables. Then for all $\lambda \vdash n$, the action of $f((J_i)_{1 \leqslant i \leqslant n})$ in V_λ is* $\dim \lambda \cdot f((c(\Box))_{\Box \in \lambda})\mathbf{Id}$.

Theorem 2.1.4 (Frobenius formula)**.** *Let $Cov(l_1, l_2, \dots, l_r; \lambda_1, \lambda_2, \dots, \lambda_\ell)$ be the number of $r+\ell$-uples of permutations $(\sigma_1, \sigma_2, \dots, \sigma_r, \phi_1, \phi_2, \dots, \phi_\ell)$ of $\mathfrak{S}_n$ such that for all i, σ_i has l_i cycles and ϕ_i has cycle type λ_i, verifying the equation*

$$\sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_r \cdot \phi_1 \cdot \phi_2 \cdot \dots \cdot \phi_\ell = \mathbf{1},$$

then

$$\begin{aligned} &\sum_{l_1, l_2, \dots, l_r \geqslant 1} Cov(l_1, l_2, \dots, l_r, \lambda_1, \lambda_2, \dots, \lambda_\ell) \prod_{i=1}^{r} u_i^{n-l_i} \\ &= \sum_{\nu \vdash n} (\dim \nu)^2 \prod_{i=1}^{r} \prod_{\square \in \nu} (1 + u_i c(\square)) \prod_{j=1}^{\ell} \frac{|C_{\lambda_i}| \chi^\nu(\lambda_i)}{\dim \nu}. \end{aligned} \tag{2.1.2}$$

Remark 2.1.5. *The number of factorisations is named Cov, because factorisations of permutations also represent ramified coverings of the sphere (see [LZ04]).*

Before getting to the proof, we introduce some more notations, and a lemma. For all λ, let $K_\lambda = \sum_{\sigma \in C_\lambda} \sigma$. It is a central element of $\mathbb{C}[\mathfrak{S}_n]$ since it is invariant under conjugation. Set also

$$P_n(u) = \sum_{\sigma \in \mathfrak{S}_n} u^{n-l(\sigma)} \sigma,$$

where $l(\sigma)$ is the number of cycles of σ.

Lemma 2.1.6. *We have the following equality:*

$$P_n(u) = \prod_{k=1}^{n} (1 + u J_k). \tag{2.1.3}$$

Also, $P_n(u)$ is a central element of $\mathbb{C}[\mathfrak{S}_n]$.

Proof. The second point is true because we have

$$P_n(u) = \sum_{\lambda \vdash n} K_\lambda u^{n-l(\lambda)}.$$

We will prove (2.1.3) by induction on n. It is easily verified for $n = 1$, now suppose it is for a given n.

We give a bijection between elements of $\mathfrak{S}_{n+1}$ on the one hand, and elements of $\mathfrak{S}_n$ along with an integer k in $\{1, 2, \dots, n+1\}$ on the other hand. Starting from $\sigma \in \mathfrak{S}_{n+1}$, we set $k = \sigma(n+1)$. If $k = n+1$, then σ' is σ restricted to $\{1, 2, \dots, n\}$. Otherwise, let σ' be the restriction of $(k, n+1)\sigma$ to $\{1, 2, \dots, n\}$. In the first case, $l(\sigma') = l(\sigma) - 1$, in the second $l(\sigma') = l(\sigma)$.

Therefore we have

$$P_{n+1}(u) = \sum_{\sigma\in\mathfrak{S}_n} u^{n-l(\sigma)}\sigma(1+uJ_{n+1}),$$

and we conclude by induction. □

We can now prove the Frobenius formula.

Proof of Theorem 2.1.4. The LHS of (2.1.2) is equal to:

$$[\mathbf{1}]\prod_{i=1}^{r}\prod_{k=1}^{n}P(u_i)\prod_{j=1}^{\ell}K_{\lambda_j}$$

where $[\mathbf{1}]$ means taking the coefficient of the identity permutation in the sum.

We can see $X=\prod_{i=1}^{r}\prod_{k=1}^{n}P(u_i)\prod_{j=1}^{\ell}K_{\lambda_j}$ as an operator on $\mathbb{C}[\mathfrak{S}_n]$ that acts by multiplication. Obviously, for all $\sigma\in\mathfrak{S}_n$, $[\mathbf{1}]X=[\sigma]X\sigma$. Thus, if we see X as a matrix, it has a constant diagonal and therefore

$$[\mathbf{1}]X=\frac{1}{n!}tr_{\mathbb{C}[\mathfrak{S}_n]}(X)^{1}.$$

Using the decomposition of $\mathbb{C}[\mathfrak{S}_n]$ in irreducible representations, we get that the LHS of (2.1.2) is equal to

$$\frac{1}{n!}\sum_{\nu\vdash n}(\dim\nu)\chi^{\nu}(X).$$

Using centrality, and by (2.1.1), the LHS of (2.1.2) is

$$\frac{1}{n!}\sum_{\nu\vdash n}\dim\nu\cdot\chi^{\nu}(1)\cdot\prod_{i=1}^{r}\frac{\chi^{\nu}\left(\prod_{k=1}^{n}P(u_i)\right)}{\dim\nu}\prod_{j=1}^{\ell}\frac{\chi^{\nu}(K_{\lambda_j})}{\dim\nu}.$$

By Proposition 2.1.3, we know that

$$\frac{\chi^{\nu}\left(\prod_{k=1}^{n}(1+u_iJ_k)\right)}{\dim\nu}=\prod_{\square\in\nu}(1+u_ic(\square)),$$

and since $\chi^{\nu}(1)=\dim\nu$, it finishes the proof. □

[1] this will be helpful since the trace is invariant by change of basis !

2.1.2 Symmetric functions

The theory of symmetric functions is a rich and active field of research, however we will only include here a brief presentation of the symmetric functions we need for our work.

Fix a countably infinite set of indeterminates $(x_1, x_2, \ldots)$. A *symmetric function* f is a formal sum of monomials in the x_i's with coefficients in $\mathbb{C}$, such that:

- f is invariant by exchanging any two of the variables,
- there exists $M > 0$ such that the total degree of every monomial in f is bounded by M.

We will describe two particular bases of the space of symmetric functions that we will make use of. Other classical bases include *elementary*, *monomial*, and *complete homogeneous* symmetric functions, we will not deal with them here, we refer to [Sta99] for a more complete exposition.

We start with the *powersums*, that are defined this way: for all j,

$$p_j = \sum_i x_i^j,$$

and for all λ, $p_\lambda = \prod_{j=1}^{l(\lambda)} p_{\lambda_j}$.

Definition 2.1.4. A *semi-standard Young tableau* (SSYT) of shape λ is constructed from the diagram of λ by placing a positive integer in each box such that the rows of the tableau are weakly increasing and the columns are strictly increasing (see Figure 2.3).

Let T be an SSYT containing exactly α_i entries equal to i for all i. Then we set $\mathbf{x}^T = \prod_i x_i^{\alpha_i}$. Then the *Schur functions* are defined as

$$s_\lambda = \sum \mathbf{x}^T,$$

where the sum spans over all SSYT's of shape λ.

It is not obvious, but not hard to see, that Schur functions are symmetric functions. In fact, they are even a basis, and their change-of-basis coefficients to the powersums involve the characters of the symmetric group.

Theorem 2.1.7 (Classical, see [Sta99])**.** *The p_λ's (resp. the s_λ's) are a basis for the ring of symmetric functions*[2]*, and, for all $\nu \vdash n$,*

$$s_\nu = \frac{1}{n!} \sum_{\lambda \vdash n} \chi^\nu(\lambda) |C_\lambda| p_\lambda.$$

[2]this is including $p_\emptyset = s_\emptyset = 1$.

$$T = \begin{array}{|c|c|c|c|} \cline{1-1} 9 & \multicolumn{3}{c}{} \\ \cline{1-1} 6 & \multicolumn{3}{c}{} \\ \cline{1-3} 4 & 4 & 5 & \multicolumn{1}{c}{} \\ \hline 2 & 2 & 2 & 3 \\ \hline \end{array} \qquad\qquad \mathbf{x}^T = x_2^3 x_3 x_4^3 x_5 x_6 x_9$$

Figure 2.3 – An SSYT and its associated monomial.

Theorem 2.1.7 ensures that the Schur functions are uniquely determined by the values of the p_λ's, which in turn are uniquely determined by the sequence $\mathbf{p} = (p_1, p_2, \ldots)$. Therefore, in what follows, we will consider the powersums as the "initial" variables, and parametrize the Schur functions by the powersums, thus writing $s_\nu(\mathbf{p})$ for the Schur function indexed by ν, expressed as a polynomial in the powersums.

2.2 The KP and 2-Toda hierarchies

In this section, we introduce the KP and 2-Toda hierarchies. We insist on one way of constructing them, but as it is mentioned, there are many more approaches. What's more, we mainly focus on the concepts and objects that we need in the context of maps, but there is much more to this story. We refer to three very good introductions to the topic: [MJD00] to understand how the KP hierarchy is built out of the original KP equation, [KR87] to see things through the lens of infinite dimensional Lie algebras, and [AZ13] for the fermionic approach. Finally, the appendix of [Oko01] sets the basics in a very compact way.

2.2.1 The semi infinite wedge space

Before getting to the hierarchies themselves, we need some more algebraic setup. This subsection is devoted to explaining the *semi infinite wedge space* and the operators that act on it.

Everything will be defined over the field of complex numbers $\mathbb{C}$, but all the results can be extended to complex formal series in a (possibly countably infinite) number of variables. In the rest of this section we will use the words "operators" and "vectors" in the sense of formal series, for a set of underlying variables. Everything is understood in the formal sense, what is important is that the coefficient of each monomial (in the underlying variables) in all vector or operator elements is well defined, and that no infinite coefficient appears in computations.

Definition 2.2.1. A *Maya diagram* (see Figure 2.4) is a decoration of $\mathbb{Z}+\frac{1}{2}$ with a particle or an antiparticle at each position, such that there exists $M > 0$ such that there are only particles at positions $< -M$ and only antiparticles at positions $> M$. The semi infinite wedge space $\Lambda^{\frac{\infty}{2}}$ is the vector space generated by the Maya diagrams. It is equipped with an inner product by making the Maya diagrams orthogonal to each other and of norm 1.

Remark 2.2.1. *In [Oko01], the space $\Lambda^{\frac{\infty}{2}}$ is noted $\Lambda^{\frac{\infty}{2}}V$ and is constructed out of (semi-infinite) wedge products of elements in a vector space V whose basis vectors are indexed by the half integers.*

We have some elementary operators on $\Lambda^{\frac{\infty}{2}}$:

Definition 2.2.2. For any $k \in \mathbb{Z}+\frac{1}{2}$, we define the *fermion operators* ψ_k and ψ_k^* (see Figure 2.4). For each Maya diagram $\mathbf{m}$, we set:

$$\psi_k\mathbf{m} = \begin{cases} 0 & \text{if } \mathbf{m} \text{ has a particle in position } k, \\ (-1)^{n_k}\widetilde{\mathbf{m}} & \text{otherwise,} \end{cases}$$
$$\psi_k^*\mathbf{m} = \begin{cases} 0 & \text{if } \mathbf{m} \text{ has an antiparticle in position } k, \\ (-1)^{n_k}\overline{\mathbf{m}} & \text{otherwise.} \end{cases}$$

where n_k is the number of particles of $\mathbf{m}$ in positions $> k$ (it is finite by definition of a Maya diagram). Also, $\widetilde{\mathbf{m}}$ is the same as $\mathbf{m}$ except there is a particle in position k, and $\overline{\mathbf{m}}$ is the same as $\mathbf{m}$ except there is an antiparticle in position k. Note that ψ_k and ψ_k^* are adjoint operators.

The *boson operators* are defined this way: for all $n \in \mathbb{Z}^*$, let

$$\alpha_n = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_{k-n}\psi_k^*.$$

Finally, the two *vertex operators* are

$$\Gamma_\pm(\mathbf{p}) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{n}\alpha_{\pm n}\right).$$

For all n, α_n and α_{-n} are adjoint, and thus so are $\Gamma_+(\mathbf{p})$ and $\Gamma_-(\mathbf{p})$.

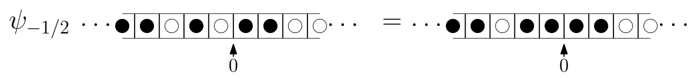


Figure 2.4 – A Maya diagram and the action of a fermion on it.

We will now define diagonal operators over $\Lambda^{\frac{\infty}{2}}$ and relate Maya diagrams to partitions.

Definition 2.2.3. We define the *normally ordered products*

$$:\psi_k\psi_k^*: := \begin{cases} \psi_k\psi_k^* & \text{if } k > 0 \\ -\psi_k^*\psi_k & \text{if } k < 0. \end{cases}$$

Note that, for a Maya diagram $\mathbf{m}$

$$:\psi_k\psi_k^*: \mathbf{m} = \begin{cases} \mathbf{m} & \text{if } k > 0 \text{ and } \mathbf{m} \text{ has a particle in position } k \\ -\mathbf{m} & \text{if } k < 0 \text{ and } \mathbf{m} \text{ has an antiparticle in position } k \\ 0 & \text{otherwise.} \end{cases}$$

The *charge operator* is:

$$C = \sum_{k\in\mathbb{Z}+\frac{1}{2}} :\psi_k\psi_k^*: .$$

The Maya diagrams are eigenvectors of C. The eigenvalue of a Maya diagram $\mathbf{m}$ is the number of particles in positive position minus the number of antiparticles in negative position. We call this number the *charge* of $\mathbf{m}$. We introduce the translation operator R: for any $\mathbf{m}$, $R\mathbf{m}$ has a particle in position $k+1$ if and only if $\mathbf{m}$ has a particle in position k. Note that if the charge of $\mathbf{m}$ is c, the charge of $R\mathbf{m}$ is $c+1$, and that the adjoint of R is R^{-1}.

There is a bijection between Maya diagrams of charge 0 and partitions: given the diagram of a partition in Russian convention, its contour can be interpreted as a walk with up- and down-steps. One obtains a Maya diagram by associating a particle to each down-step and an antiparticle to each up-step (see Figure 2.5).

We will use the braket notation, and denote the Maya diagram of charge 0 corresponding to the empty partition by $|\emptyset\rangle$, and set $|\emptyset_n\rangle = R^n|\emptyset\rangle$. We will also set $|\lambda\rangle$ to be the Maya diagram of charge 0 corresponding to the partition λ.

Finally, we define the *energy operator*

$$H = \sum_{k\in\mathbb{Z}+\frac{1}{2}} k :\psi_k\psi_k^*: .$$

In particular, $H|\lambda\rangle = |\lambda|\,|\lambda\rangle$, where $|\lambda|$ is the number of boxes in λ.

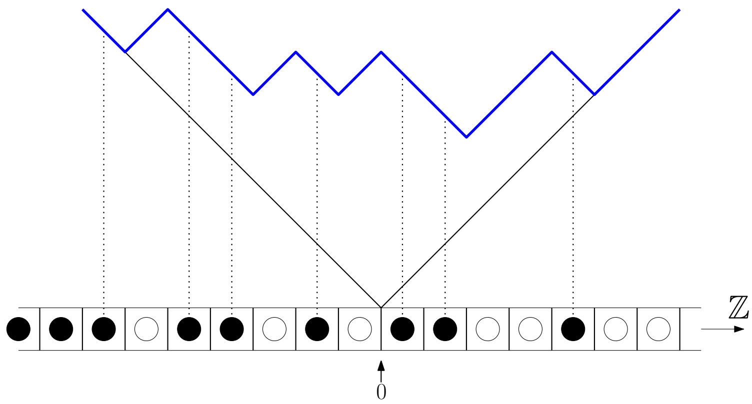


Figure 2.5 – A Maya diagram of charge 0 and its associated partition. The contour of the partition is in blue and fat.

We will need the following property later:

Proposition 2.2.2.

$$\Gamma_-(\mathbf{q})\,|\emptyset\rangle = \sum_\nu s_\nu(\mathbf{q})\,|\nu\rangle \quad \textit{and} \quad \langle\emptyset|\,\Gamma_+(\mathbf{p}) = \sum_\nu s_\nu(\mathbf{p})\,\langle\nu|\,.$$

Proof. We will only include a sketch of proof, leaving the details to initiated or motivated readers.

First, note that the action of $\psi_{k+n}\psi_k^*$ is to make a particle "jump" n slots to the right. On the associated partition, it is not hard to see that this is equivalent to adding a border strip. Therefore we can describe the action of the bosons on partitions : we have

$$\alpha_{-n}\,|\lambda\rangle = \sum_\mu (-1)^{ht(\mu\setminus\lambda)}\,|\mu\rangle$$

where the sum spans over all μ such that $\mu \setminus \lambda$ is a border strip. Therefore, expanding $\Gamma_-(\mathbf{q})\,|\emptyset\rangle$ in the q_λ's for all partitions λ, one notices that it is possible to apply the Murnaghan–Nakayama rule (Theorem 2.1.2) to write it as a sum over the q_λ's weighted by characters.

Applying Theorem 2.1.7 (change of basis between powersums and Schur functions) finishes the proof. □

We end this section by stating the *boson-fermion correspondence* (a proof can be found in [KR87]): the boson operators are defined in terms of the fermions, but we can go the other way and express the fermions in terms of the bosons. More precisely, let

$$\psi(z) = \sum_{i \in \mathbb{Z}+\frac{1}{2}} z^i \psi_i, \quad \psi^*(z) = \sum_{i \in \mathbb{Z}+\frac{1}{2}} z^{-i} \psi_i^* \tag{2.2.1}$$

be the "generating Laurent series of the fermions", then

$$\begin{aligned} \psi(z) &= z^C R \Gamma_-(\{z\}) \Gamma_+ \left(-\left\{z^{-1}\right\}\right), \\ \psi^*(z) &= R^{-1} z^{-C} \Gamma_-(-\{z\}) \Gamma_+ \left(\left\{z^{-1}\right\}\right), \end{aligned} \tag{2.2.2}$$

where $\{z\} = (z, z^2, z^3, \ldots)$.

2.2.2 The space of solutions

There are several definitions of the hierarchies in the literature. Here, we will give the most general definition, in which, to any operator satisfying a certain bilinear identity, one associates a formal power series which satisfies certain bilinear equations.

Remark 2.2.3. *Here, we construct the space of solutions, and then derive the equations they satisfy. It might be suprising to see things defined this way, but it greatly simplifies the exposition, and it is coherent with the proofs (see Proposition 2.2.4). In [MJD00], things are presented in a more "natural" way: an ordered set of equations (a hierarchy) is constructed out of the original KP equation, and the space of solutions is determined afterwards.*

We need yet another operator, let:

$$\Omega = \sum_k \psi_k \otimes \psi_k^*.$$

A *tau-function* of the KP hierarchy is a function of the form

$$\tau(\mathbf{p}) = \langle \emptyset | \Gamma_+(\mathbf{p}) A | \emptyset \rangle, \tag{2.2.3}$$

where A is an operator satisfying

$$[A \otimes A, \Omega] = 0, \tag{2.2.4}$$

where $[,]$ is the ring commutator.

After some algebraic manipulations on (2.2.3) and (2.2.4) (using in particular the boson-fermion correspondence (2.2.2)), we have that, for every τ that is a tau-function of the KP hierarchy, the following equation holds:

$$[t^{-1}]\exp\left(\sum_{k\geqslant 1}\frac{t^k}{k}(p_k-q_k)\right)\exp\left(-\sum_{k\geqslant 1}t^{-k}(\frac{\partial}{\partial p_k}-\frac{\partial}{\partial q_k})\right)\tau(\mathbf{p})\tau(\mathbf{q})=0. \tag{2.2.5}$$

If τ is a tau-function of the KP hierarchy, then $F=\log\tau$ is said to be a *solution* of the KP hierarchy. Each coefficient in the Taylor expansion of (2.2.5) with respect to the q_i's, corresponds to a PDE satisfied by F (in the variables $\mathbf{p}$). For instance the "original KP equation" reads:

$$F_{3,1}=F_{2,2}+\frac{1}{2}F_{1,1}^2+\frac{1}{12}F_{1,1,1,1} \tag{2.2.6}$$

where indices indicate partial derivatives, for instance $F_{3,1}=\frac{\partial^2}{\partial p_1\partial p_3}F$.

A family of functions (τ_n) is said to be a family of tau-functions of the 2-*Toda hierarchy* if they can be written as

$$\tau_n(\mathbf{p},\mathbf{q})=\langle\emptyset_n|\Gamma_+(\mathbf{p})A\Gamma_-(\mathbf{q})|\emptyset_n\rangle\,,$$

with A satisfying (2.2.4).

In that case, $\tau:=\tau_0$ is also a tau-function of the KP hierarchy, if we treat the q_i's as mute variables (one can check that if A satisfies (2.2.4), then so does $A\Gamma_-(\mathbf{q})$). It is in that sense that the 2-Toda hierarchy is a generalization of the KP hierarchy.

Again, algebraic computations lead to equations. Any family (τ_n) of tau-functions of the 2-Toda hierarchy satisfies the following equation for all $n,m\in\mathbb{Z}$:

$$\begin{aligned}\left[t^{n-m}\right]\gamma\left(\frac{1}{t},-2\mathbf{s}'\right)\tau_{m+1}\left(\mathbf{p}+\mathbf{s},\mathbf{q}+\mathbf{s}'+\{t\}\right)\tau_n\left(\mathbf{p}-\mathbf{s},\mathbf{q}-\mathbf{s}'-\{t\}\right)=\\ \left[t^{m-n}\right]\gamma\left(\frac{1}{t},2\mathbf{s}\right)\tau_m\left(\mathbf{p}+\mathbf{s}-\{t\},\mathbf{q}+\mathbf{s}'\right)\tau_{n+1}\left(\mathbf{p}-\mathbf{s}+\{t\},\mathbf{q}-\mathbf{s}'\right)\end{aligned} \tag{2.2.7}$$

with $\gamma\left(t,\mathbf{s}\right)=\exp\left(\sum_{n\geqslant 1}\frac{s_n}{n}t^n\right)$.

And once again, with the help of Taylor expansion, we can derive differential equations between the τ_n's. The simplest one (and the one we will make use of) is

$$\frac{\partial^2}{\partial p_1\partial q_1}\log\tau_0=\frac{\tau_1\tau_{-1}}{\tau_0^2}. \tag{2.2.8}$$

As stated in [GJ08][3], a large family of operators acting diagonally on partitions gives rise to tau functions.

Proposition 2.2.4 (Content product solutions). *Let f be a function with positive values. Then*

$$A = \exp\left(\sum_{k\in\mathbb{Z}+\frac{1}{2}} \operatorname{sgn}(k)\left(\frac{1}{2}\log f(0) + \sum_{i=1}^{|k|-1/2}\log f(\operatorname{sgn}(k)\cdot i)\right) : \psi_k\psi_k^* :\right)$$

is well defined and satisfies (2.2.4). The associated tau-function τ is called a content product solution *because then*

$$\tau = \sum_{\nu}\left(\prod_{\square\in\nu} f(c(\square))\right) s_\nu(\mathbf{p})s_\nu(\mathbf{q}).$$

Proof. The second point comes from the fact that

$$A\left|\lambda\right\rangle = \left(\prod_{\square\in\lambda} f(c(\square))\right)\left|\lambda\right\rangle,$$

which can be observed by the correspondence between Maya diagrams and partitions. For the first, let

$$F = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \operatorname{sgn}(k)\left(\frac{1}{2}\log f(0) + \sum_{i=1}^{|k|-1/2}\log f(\operatorname{sgn}(k)\cdot i)\right) : \psi_k\psi_k^* :$$

so that $A = e^F$. One can check by hand that for all k and all complex numbers c, $1\otimes c : \psi_k\psi_k^* :$ satisfies (2.2.4). It is a general fact that if $[X,Y]=0$ then $[X,e^Y]=0$, therefore $\exp(1\otimes c : \psi_k\psi_k^*)$ satisfies (2.2.4) as well.

By taking the product (and carefully choosing c each time), we find that $\exp(1\otimes F)$ satisfies (2.2.4), and we conclude by noticing that

$$A\otimes A = \exp(1\otimes F)\cdot\exp(F\otimes 1).$$

□

2.2.3 Other definitions

Now we give two other ways of defining the KP hierarchy[4]. They are a little more restrictive than the one we gave in the previous section (the space of solutions is smaller), but they are still very important. Indeed, they describe a wide space of solutions, and even all the solutions under natural, well-chosen, restrictions. The proofs can be found in [MJD00] and [AZ13].

[3] it can also easily be deduced from [Oko00].
[4] things would be similar for the 2-Toda hierarchy.

The Sato Grassmanian. First, one can define the solutions of the KP hierarchy as points in a certain orbit under the action of a Lie algebra. In 1981, Sato [Sat81] introduced an infinite dimensional Grassmanian.

Definition 2.2.4 (The Sato Grassmanian)**.** We consider infinite dimensional matrices (with shifted indices) $M = (m_{i,j})_{i,j \in \mathbb{Z}+\frac{1}{2}}$ satisfying

$$\exists N \text{ such that } |i-j| > N \Rightarrow a_{i,j} = 0. \tag{2.2.9}$$

To such a matrix M, we can associate an operator on $\Lambda^{\frac{\infty}{2}}$

$$X_M = \sum_{i,j} m_{i,j} : \psi_{-i} \psi_j^* : .$$

Then the *Sato Grassmanian* is defined as

$$\mathfrak{gl}(\infty) = \{X_M | A \text{ satisfies } (2.2.9)\} \oplus \mathbb{C}$$

The Sato Grassmanian $\mathfrak{gl}(\infty)$ is obviously an additive group, but one can check that it is also a Lie algebra. Note that $\Gamma_{\pm}(\mathbf{p}) \in \mathfrak{gl}(\infty)$ (in the sense of formal series). It is proved that all the matrices A of the form

$$A = e^{X_1} e^{X_2} \dots e^{X_k} \text{ with } X_i \in \mathfrak{gl}(\infty)$$

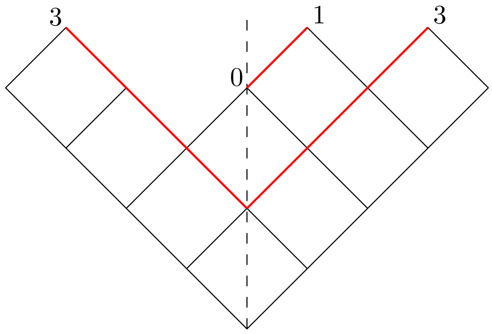
for finite k satisfy the commutation relation (2.2.4). Therefore, one way to see things is that the (interesting) solutions of the KP hierarchy are the orbit of 1 under the action of $\mathfrak{gl}(\infty)$.

The Plücker relations. Since matrices are involved, it will not be so surprising to see that the KP hierarchy can be defined in a determinantal way. Consider a big matrix, and now calculate the determinants of all its submatrices of a given small size. Obviously, since the submatrices overlap, the determinants are not independent. The *Plücker relations* give the algebraic relations between these determinants. Here we will consider the simplest version of it, and write the Plücker relations in the language of partitions.

We first need to introduce the *Frobenius coordinates* of a partition (which have lots in common with the Maya diagram presentation). Let λ be a partition, and let $\ell = \max\{i | \lambda_i - i \geqslant 0\} = \max\{i | \lambda'_i - i \geqslant 0\}$ (it is not hard to check that the last two quantities are the same). Then we write $\lambda = (\vec{\alpha}|\vec{\beta})$ where $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$, $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_\ell)$ and

$$\alpha_i = \lambda_i - i \quad \text{and} \quad \beta_i = \lambda'_i - i.$$

The Frobenius coordinates determine the partition uniquely (see Figure 2.6 for an example), and conversely, for all ℓ and all pairs of strictly decreasing ℓ-uples of nonnegative integers $\vec{\alpha}$ and $\vec{\beta}$, there exists a partition whose Frobenius coordinates are $(\vec{\alpha}|\vec{\beta})$.



$$\vec{\alpha} = (3,1)$$
$$\vec{\beta} = (3,0)$$

Figure 2.6 – A partition and its Frobenius coordinates.

Definition 2.2.5 (Plücker relations (see [AZ13], eq. 3.36))**.** We say that a family c_λ indexed by partitions satisfies the Plücker relations if for all $\lambda = (\vec{\alpha}|\vec{\beta})$, we have, for all r, s:

$$c_{(\vec{\alpha}|\vec{\beta})} c_{\left(\vec{\alpha}_{\cancel{r}\cancel{s}}|\vec{\beta}_{\cancel{r}\cancel{s}}\right)} = c_{\left(\vec{\alpha}_{\cancel{r}}|\vec{\beta}_{\cancel{r}}\right)} c_{\left(\vec{\alpha}_{\cancel{s}}|\vec{\beta}_{\cancel{s}}\right)} - c_{\left(\vec{\alpha}_{\cancel{r}}|\vec{\beta}_{\cancel{s}}\right)} c_{\left(\vec{\alpha}_{\cancel{s}}|\vec{\beta}_{\cancel{r}}\right)} \tag{2.2.10}$$

with

$$\begin{aligned} \left(\vec{\alpha}_{\cancel{j}}|\vec{\beta}_{\cancel{k}}\right) &= \left(\alpha_1, \ldots, \cancel{\alpha_j}, \ldots, \alpha_\ell | \beta_1, \ldots, \cancel{\beta_k}, \ldots, \beta_\ell\right), \\ \left(\vec{\alpha}_{\cancel{r},\cancel{s}}|\vec{\beta}_{\cancel{r},\cancel{s}}\right) &= \left(\alpha_1, \ldots, \cancel{\alpha_r}, \ldots, \cancel{\alpha_s}, \ldots, \alpha_\ell | \beta_1, \ldots, \cancel{\beta_r}, \ldots, \cancel{\beta_s}, \ldots, \beta_\ell\right). \end{aligned}$$

Proposition 2.2.5. *A function $\tau(\mathbf{p})$ that can be expanded as*

$$\tau(\mathbf{p}) = \sum_\lambda a_\lambda s_\lambda(\mathbf{p})$$

such that the a_λ satisfy the Plücker relations is a tau-function of the KP hierarchy.

Remark 2.2.6. *The quadratic nature of (2.2.10) is coherent with the fact that the KP hierarchy gives quadratic equations on τ (see (2.2.5)).*

2.3 Maps and 2-Toda hierarchy

In all generality, let us introduce a generating function of constellations, that will be shown to satisfy the 2-Toda hierarchy (and therefore the KP hierarchy as well). Since maps are a particular instance of constellations, we will also be able to derive formulas for maps by specializing the generating function of constellations to maps.

Fix an integer r. We introduce the following generating function of (labelled, non-necessarily connected) constellations, in accordance with the representation of constellations as factorisations of permutations (see Section 1.1.4, and Theorem 2.1.4 for a definition of the coefficients Cov):

$$\tau(z, \mathbf{p}, \mathbf{q}, (u_j)) = \sum_{\substack{n\geqslant 0\\ |\mu|=|\lambda|=n\\ l_i\geqslant 1\ \forall i}} \frac{z^n}{n!}\prod_{i=1}^{r} u_i^{n-l_i} p_\lambda q_\mu Cov(l_1, l_2, \ldots, l_r; \lambda, \mu). \tag{2.3.1}$$

Remark 2.3.1. *These $(r+2)$-uples represent $(r+1)$-constellations, where we control the degree distributions of faces and vertices of color $r+1$, while controlling the number of vertices of each other color. However, in most cases we will apply the specialization $\mu = 1^n$ (and sometimes even $\lambda = 1^n$ as well), and thus they will either represent r- or $(r-1)$-constellations. Indeed setting $\mu = 1^n$ forces one of the permutations to be the identity, and we truly have only an $(r+1)$-uple.*

It is a classical result (under different forms and variants, see for instance [GJ08, Oko00]) that the function τ can be expressed in terms of elements and operators of $\Lambda^{\frac{\infty}{2}}$:

Lemma 2.3.2.

$$\tau(z, \mathbf{p}, \mathbf{q}, (u_j)) = \langle \emptyset | \Gamma_+(\mathbf{p}) z^H \Lambda \Gamma_-(\mathbf{q}) | \emptyset \rangle \tag{2.3.2}$$

with $\Lambda = \prod_{j=1}^{r} \exp(F(u_j))$ *and*

$$F(u) = \sum_{k>0}\sum_{i=0}^{k-1/2} \log(1+ui)\psi_k\psi_k^* + \sum_{k<0}\sum_{i=0}^{-k-1/2} \log(1-ui)\psi_k^*\psi_k.$$

Proof. By the Frobenius formula (Theorem 2.1.4), we have

$$\tau(z, \mathbf{p}, \mathbf{q}, (u_j)) =$$
$$\sum_{\substack{|\lambda|=|\mu|=n>0\\ l>0}} \frac{z^n}{(n!)^2} p_\lambda q_\mu \sum_{|\nu|=n} \frac{1}{n!} \prod_{j=1}^{r}\left(\prod_{\Box\in\nu}(1+u_j c(\Box))\right) |C_\lambda|\chi^\nu(\lambda)|C_\mu|\chi^\nu(\mu).$$

Next, thanks to Theorem 2.1.7, we can express τ in terms of Schur functions.

$$\tau(z, \mathbf{p}, \mathbf{q}, (u_j)) = \sum_{\substack{|\nu|=n>0\\ l>0}} z^n s_\nu(\mathbf{p}) s_\nu(\mathbf{p}) \prod_{j=1}^{r}\left(\prod_{\Box\in\nu}(1+u_j c(\Box))\right).$$

One concludes the proof by using Proposition 2.2.2. $\Box$

Lemma 2.3.2 together with Proposition 2.2.4 show that τ, the generating functions of constellations that we introduced is a tau-function of the 2-Toda hierarchy, if we introduce the τ_n as $\tau_n(z, \mathbf{p}, \mathbf{q}, (u_j)) = \langle \emptyset_n | \Gamma_+(\mathbf{p}) z^H \Lambda \Gamma_-(\mathbf{q}) | \emptyset_n \rangle$ and we set $\tau = \tau_0$.

The equations for the series of constellations obtained from the hierarchies are still not fully understood combinatorially (hence the search for bijections, to which Chapter 4 is a contribution). However, they are very powerful: they gave us several efficient counting formulas (the latest ones can be found in Chapter 5), which proved to be useful even for probabilistic considerations (see Chapters 6 and 7).

Chapter 3

Peeling, local limits and infinite planar maps

This chapter is devoted to introducing the objects that we will show to be the local limits of high genus random maps. In order to keep things light, we will mostly focus on the concepts and results that will be used in Chapters 6 and 7. For an introduction to hyperbolic triangulations of the plane (without loops), one can read [Cur16], for general triangulations and hyperbolic bipartite maps there is [Bud18a]. Many geometric properties of infinite maps are not dealt with here. For a detailed exposition (in the case of bipartite maps), read [Cur19]. Also, the proofs of the results exposed in this chapter can be found there.

3.1 Triangulations

3.1.1 An alternative definition of local limits

In the introduction, we defined the local convergence as the convergence (in distribution) with respect to a certain topology on the space of (finite and infinite) maps. Here we give a second definition, based on the convergence of the law of the neighbourhood of the root, that is equivalent but also easier to understand, and that is the one that is used in practice.

Before that, we need to define the notion of *map inclusion*. Here we will stick with triangulations, but the definitions are more or less the same in the general case.

Definition 3.1.1. Let t be a finite planar triangulation with a simple boundary of size p, and let T be a triangulation (not necessarily finite). We say

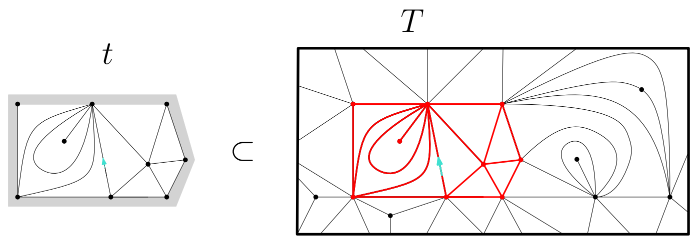


Figure 3.1 – A small triangulation t is included in a bigger one T. Only the neighbourhood of the root of T is represented here.

that

$$t \subset T$$

if there exists a triangulation $\widetilde{T}$ with a simple hole of size p, such that one can fill the hole of $\widetilde{T}$ with t and obtain T.

In other words, it means that by carving a finite neighbourhood of T, one can obtain t. Alternatively, it is possible to superimpose t on T such that both roots coincide.

We can now give a definition of local convergence in terms of map inclusion. To make things easier, we will only consider convergence towards infinite maps of the plane.

Definition 3.1.2. If (T_n) is a sequence of random triangulations, and $\mathbb{T}$ is a random infinite triangulation of the plane, we say that (T_n) converges locally in distribution towards $\mathbb{T}$ if for all finite planar triangulations t with a simple boundary

$$P(t \subset T_n) \to P(t \subset \mathbb{T})$$

as $n \to \infty$.

It is not hard to see that this definition is equivalent to the one given in Chapter 1.

3.1.2 The UIPT

Let T_n be a uniform triangulation of the sphere with $2n$ triangles. In 2002, Angel and Schramm [AS03] showed the following result:

Theorem 3.1.1. *The sequence (T_n) converges locally (in distribution) to a random infinite triangulation of the plane T_∞.*

The triangulation T_∞ is called the *Uniform Infinite Planar Triangulation* (UIPT). It is uniquely determined by the law of the neighbourhood of the root, given by the following proposition:

Proposition 3.1.2. *Let t be a finite planar triangulation with a simple boundary of size p (its* perimeter*) and v vertices (its* volume*). Then*

$$\mathbb{P}(t \subset T_\infty) = C_p \lambda_c^v$$

where the C_p are constants that do not depend on t and $\lambda_c = \frac{1}{12\sqrt{3}}$ is the radius of convergence of the generating function of planar triangulations.

The UIPT has many more properties. First, it is *one-ended*, namely if one removes a finite part of T_∞, then the remaining map has only one infinite connected component. It is also *reversible* (its distribution is invariant under reversing the orientation of the root edge) and *stationary* (invariant under rerooting at the first step of the simple random walk). These properties follow from general rerooting invariance properties of uniform finite triangulations.

3.1.3 Hyperbolic triangulations of the plane

Seeing Proposition 3.1.2, one might ask if, more generally, for $\lambda \in \mathbb{R}^+$, there exists a random triangulation of the plane $\mathbb{T}_\lambda$ such that

$$\mathbb{P}(t \subset \mathbb{T}_\lambda) = C_p \lambda^{|t|} \tag{3.1.1}$$

for all finite triangulations with a simple boundary of size p. In other words, can we replace λ_c with something else (provided that we change the value of the constants C_p) ? Property (3.1.1) is called the *spatial Markov property*, because if t is a neighbourhood of the root of $\mathbb{T}_\lambda$, then the law of $\mathbb{T}_\lambda \setminus t$ (the "future") depends only on the perimeter of t (the "present").

Curien answered this question in [Cur16] : for all $0 < \lambda \leqslant \lambda_c$, there exists a unique set of constants (C_p) and a unique (in distribution) random infinite triangulation $\mathbb{T}_\lambda$ satisfying the spatial Markov property. This family of maps is called the *Planar Stochastic Hyperbolic Infinite Triangulations* (PSHT). It is parametrized by a single real λ, and for $\lambda = \lambda_c$, we recover the UIPT. This follows a similar work on infinite triangulations of the half plane [AR15]. The work of Curien originally focused on triangulations without loops, it has been adapted to general triangulations by Budzinski in [Bud18a].

Peeling the PSHT. The PSHT can be "explored" by a random process named the *peeling process*. The principle is to discover the map step by step, triangle by triangle. In fact, there are many peeling processes, but they all share some key features. Given an instance $\mathbb{T}_\lambda$ of the PSHT, we can describe a sequence of random finite planar triangulations with a simple boundary

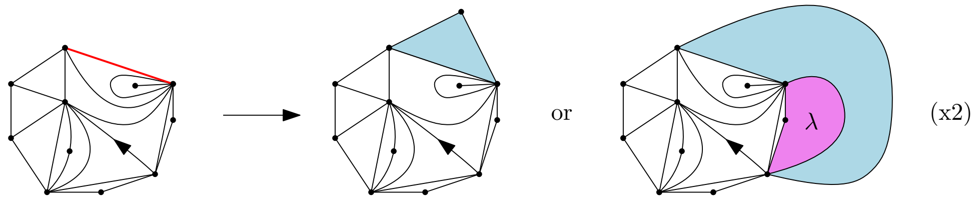


Figure 3.2 – A peeling step. The unexplored part is on the outside. When the new triangle cuts the unexplored part in two, the finite part is distributed as a Boltzmann triangulation of parameter λ.

that is strictly increasing (with respect to map inclusion), all of them being included in $\mathbb{T}_\lambda$.

A *peeling algorithm* $\mathcal{A}$ takes as input a finite triangulation t with a simple boundary, and outputs an edge $\mathcal{A}(t)$ on the boundary of t. Given a peeling algorithm $\mathcal{A}$, we can define an increasing sequence $\left(\mathcal{E}_\lambda^{\mathcal{A}}(k)\right)_{k\geqslant 0}$ of triangulations with a simple boundary such that $\mathcal{E}_\lambda^{\mathcal{A}}(k) \subset \mathbb{T}_\lambda$ for every k in the following way:

- the map $\mathcal{E}_\lambda^{\mathcal{A}}(0)$ is the trivial map consisting of the root edge only,
- for every $k \geqslant 1$, the triangulation $\mathcal{E}_\lambda^{\mathcal{A}}(k+1)$ is obtained from $\mathcal{E}_\lambda^{\mathcal{A}}(k)$ by adding the triangle incident to $\mathcal{A}\left(\mathcal{E}_\lambda^{\mathcal{A}}(k)\right)$ outside of $\mathcal{E}_\lambda^{\mathcal{A}}(k)$ and, if this triangle creates a finite hole, all the triangles in this hole (see Figure 3.2).

Such an exploration is called *filled-in*, because all the finite holes are filled at each step. Peeling processes can also be defined for all infinite planar triangulations, and even for finite triangulations (with some precautions).

For the PSHT, we denote by $P^\lambda(k)$ and $V^\lambda(k)$ the perimeter and volume of $\mathcal{E}_\lambda^{\mathcal{A}}(k)$. The one-endedness of the PSHT guarantees that $\mathcal{E}_\lambda^{\mathcal{A}}(k)$ only has one boundary. The spatial Markov property ensures that $\left(P^\lambda(k), V^\lambda(k)\right)_{k\geqslant 0}$ is a Markov chain on $\mathbb{N}^2$ and that its transitions do not depend on the algorithm $\mathcal{A}$. The transition probabilities for $P^\lambda(k)$ determine the probabilities of each case of Figure 3.2, and they can be calculated thanks to the spatial Markov property. In case the peeling step creates a finite hole of size p, the finite triangulation that fills this hole is sampled according to a Boltzmann distribution of parameter λ.

We know the asymptotic behaviour of the perimeter and volume processes. In particular, we have [Cur16]:

$$\frac{P^\lambda(k)}{V^\lambda(k)} \xrightarrow[k\to+\infty]{a.s.} 1-4h, \tag{3.1.2}$$

where h is uniquely determined by the following equation:

$$\lambda = \frac{h}{(1+8h)^{3/2}}.$$

Remark 3.1.3. *For $\lambda < \lambda_c$, this means that the perimeter grows linearly in the volume, which is a typical hyperbolic behaviour. For $\lambda = \lambda_c$, we have $h = 1/4$, and the perimeter grows way more slowly than the volume.*

3.2 Bipartite maps

3.2.1 Peeling and inclusion in bipartite maps

All the concepts and objects defined in Section 3.1 can be transposed, and in some sense, generalized, to bipartite maps. To make things easier, we need to define slightly more different notions of map inclusion and peeling, as defined in [Bud15].

Map inclusion for bipartite maps. Given a bipartite map M, let M^* be its dual map. Let $\mathfrak{e}$ be a connected subset of edges of M^* such that the root vertex of M^* is incident to $\mathfrak{e}$. To $\mathfrak{e}$, we associate the map $\mathfrak{m}$ that is obtained by gluing the faces of M corresponding to the vertices indicident to $\mathfrak{e}$ along the (dual) edges of $\mathfrak{e}$ (see Figure 3.3). Note that $\mathfrak{m}$ is a map with simple boundaries.

Now, if m is a finite bipartite map with simple holes, and M a (finite or infinite) map, we say that

$$m \subset M$$

if m can be obtained from M by the procedure described above, or if m is the map with only one edge, or if m is the map with a simple boundary of size $2p$ and one face of degree $2p$, where $2p$ is the degree of the root face in M.

With this definition of inclusion, we can adapt the alternative definition of local convergence (Definition 3.1.2) to bipartite maps almost verbatim.

The lazy peeling process of bipartite planar maps. A *peeling algorithm* is a function $\mathcal{A}$ that takes as input a finite planar bipartite map m with boundaries, and that outputs an edge $\mathcal{A}(m)$ on the boundary of m.

Given an infinite planar bipartite map M and a peeling algorithm $\mathcal{A}$, we can define an increasing sequence $\left(\mathcal{E}_M^{\mathcal{A}}(k)\right)_{k\geqslant 0}$ of bipartite maps with holes such that $\mathcal{E}_M^{\mathcal{A}}(k) \subset M$ for every k in the following way. First, the map $\mathcal{E}_M^{\mathcal{A}}(0)$

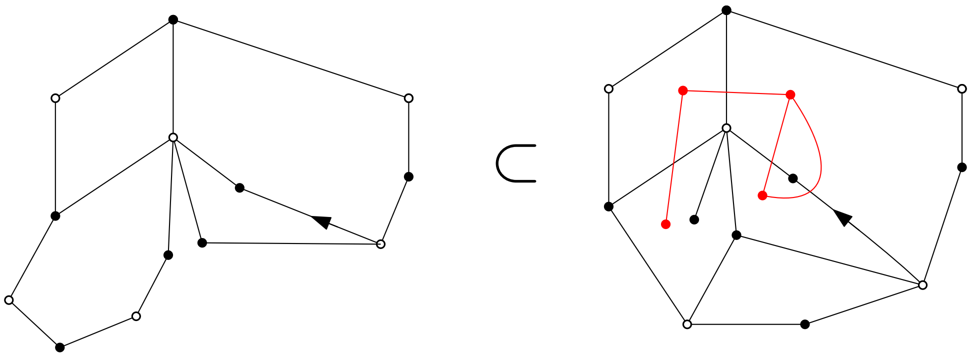


Figure 3.3 – Inclusion of bipartite maps, on an example.

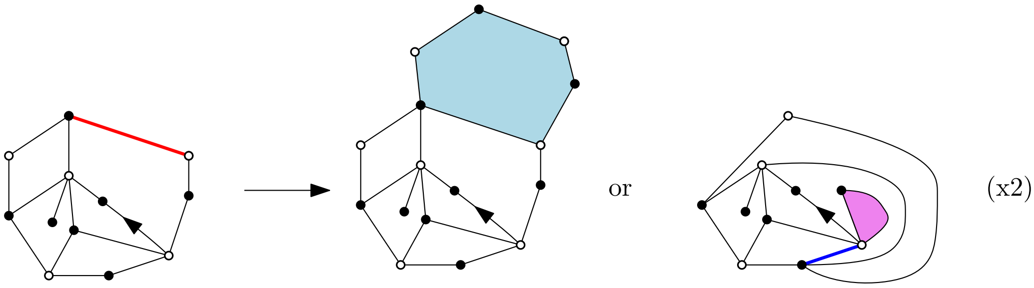


Figure 3.4 – The lazy peeling on an example. The chosen edge is in red. Either a new face is discovered (center), or the chosen edge is glued to another edge of the boundary (right, in blue).

is the trivial map consisting of the root edge only. For every $k \geqslant 1$, let F_k be the face (of M) on the other side of $\mathcal{A}(\mathcal{E}_M^{\mathcal{A}}(k))$. There are two possible cases (see Figure 3.4):

- either F_k doesn't belong to $\mathcal{E}_M^{\mathcal{A}}(k)$, then $\mathcal{E}_M^{\mathcal{A}}(k+1)$ is the map obtained from $\mathcal{E}_M^{\mathcal{A}}(k)$ by gluing a simple face of size $\deg F_k$ to $\mathcal{A}(\mathcal{E}_M^{\mathcal{A}}(k))$,
- or F_k belongs to $\mathcal{E}_M^{\mathcal{A}}(k)$. In that case, it means that there exists an edge e_k on the same boundary as $\mathcal{A}(\mathcal{E}_M^{\mathcal{A}}(k))$ such that those two edges are actually identified in M. The map $\mathcal{E}_M^{\mathcal{A}}(k+1)$ is obtained from $\mathcal{E}_M^{\mathcal{A}}(k)$ by gluing $\mathcal{E}_M^{\mathcal{A}}(k)$ and e_k together, and if this creates a finite hole, by filling it (such that the resulting map is included in M, there is only one possible way of doing so).

Such an exploration is called *filled-in*, because all the finite holes are filled at each step.

3.2.2 Hyperbolic bipartite maps of the plane

We can construct a class of infinite bipartite maps of the plane in a similar fashion as with triangulations.

Definition 3.2.1. For $\mathbf{q} = (q_i)_{i \geqslant 1}$ a collection of nonnegative real numbers, we say than a random, bipartite, planar, infinite bipartite map M is $\mathbf{q}$-Markovian, if there exists constants C_p such that for any planar bipartite map m with a simple boundary of size $2p$, we have

$$\mathbb{P}(m \subset M) = C_p \prod_{f \in m} q_{\deg f/2}$$

where the product spans over all inner faces of m.

Of course, not all sequences $\mathbf{q}$ give a valid map, but we know that if the map exists, it is unique (in law). We call this family of maps the IBPMs, for *Infinite Boltzmann Planar Maps*. There is a necessary condition about the sequence $\mathbf{q}$ for such a map to exist. If, for

$$f_{\mathbf{q}}(x) = \sum_{k \geqslant 1} q_k \binom{2k-1}{k-1} x^{k-1},$$

the equation

$$f_{\mathbf{q}}(x) = 1 - \frac{1}{x} \tag{3.2.1}$$

has a positive solution, then $\mathbf{q}$ is said to be *admissible*. If $\mathbf{q}$ is not *admissible*, then a $\mathbf{q}$-Markovian map does not exist. Furthermore, let $W(\mathbf{q})$ be the generating function of finite bipartite planar maps (a proper definition can be found in Chapter 7), then $W(\mathbf{q})$ converges if and only if $\mathbf{q}$ is admissible.

The sequence $\mathbf{q}$ also contains information about the hyperbolicity of the infinite map it parametrizes. Let $Z_{\mathbf{q}}$ be the smallest solution of (3.2.1). If $f'_{\mathbf{q}}(Z_{\mathbf{q}}) = \frac{1}{Z_{\mathbf{q}}^2}$, then $\mathbf{q}$ is said to be *critical*. If $\mathbf{q}$ is both admissible and critical, then the $\mathbf{q}$-Markovian map exists and is the local limit of a model of planar bipartite maps. Otherwise, if $f'_{\mathbf{q}}(Z_{\mathbf{q}}) < \frac{1}{Z_{\mathbf{q}}^2}$, then $\mathbf{q}$ is *subcritical*. In this case, it is possible to "measure the hyperbolicity" of the corresponding map. Set $g_{\mathbf{q}} = 4Z_{\mathbf{q}}$, let $W_p(\mathbf{q})$ be the generating function of planar bipartite maps with a boundary of size $2p$ for all p, and

$$\nu(i) = \begin{cases} q_{i+1} g_{\mathbf{q}}^i & \text{if } i \geqslant 0, \\ 2W_{-1-i}(\mathbf{q}) g_{\mathbf{q}}^i & \text{if } i \leqslant -1. \end{cases}$$

For a sequence $\mathbf{q}$ that is admissible and subcritical, a $\mathbf{q}$-Markovian map exists if and only if the equation

$$\sum_{i \in \mathbb{Z}} \nu(i) \omega^i = 1$$

has a solution $\omega > 1$. In that case, ω is in some sense the hyperbolicity parameter of the map.

Remark 3.2.1. *It is not clear if every subcritical admissible sequence* $\mathbf{q}$ *parameterizes an existing infinite map*[1]*. In Chapter 7 (Proposition 7.1.3) we make some progress on this question.*

Peeling the IBPM. As with the PSHT, the peeling process on the IBPM has many nice properties. In particular, it allows us to "read" the parameters of the IBPM. Given an instance $\mathbb{M}_{\mathbf{q}}$ of the IBPM, and a peeling algorithm $\mathcal{A}$, let P_n and V_n denote respectively the half-perimeter and total number of edges of $\mathcal{E}^{\mathcal{A}}_{\mathbb{M}_{\mathbf{q}}}(n)$.

Proposition 3.2.2. *We have the following almost sure convergences:*

$$\frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}_{P_{i+1}-P_i=j-1} \xrightarrow[n\to+\infty]{a.s.} (g_{\mathbf{q}}\omega)^{j-1}\, q_j =: r_j(\mathbf{q}) \tag{3.2.2}$$

for every $j \geqslant 1$*, and*

$$\frac{V_n - 2P_n}{n} \xrightarrow[n\to+\infty]{a.s.} \frac{\left(\sqrt{\omega}-\sqrt{\omega-1}\right)^2}{2\sqrt{\omega(\omega-1)}} =: r_\infty(\mathbf{q}). \tag{3.2.3}$$

Moreover, the weight sequence $\mathbf{q}$ *is a measurable function of the numbers* $r_j(\mathbf{q})$ *for* $j \in \mathbb{N}^* \cup \{\infty\}$*.*

We will give a proof of this property in Chapter 7. Note that $r_j(\mathbf{q})$ measures the proportion of faces of size $2j$ encountered during a peeling exploration, while $r_\infty(\mathbf{q})$ measures the relative perimeter and volume growth, and thus the hyperbolicity.

[1] in [Bud18a], there is numerical evidence supporting a negative answer to this question.

Chapter 4

Bijections for KP-based formulas

Abstract. This chapter is adapted from the article *A new family of bijections for planar maps*, published in *Journal of Combinatorial Theory, Series A* [Lou19a]. We present bijections for the planar cases of two counting formulas on maps that arise from the KP hierarchy (Goulden–Jackson and Carrell–Chapuy formulas), relying on a "cut-and-slide" operation. This is the first time a bijective proof is given for quadratic map-counting formulas derived from the KP hierarchy. Up to now, only the linear one-faced case was known (Harer–Zagier recurrence and Chapuy–Féray–Fusy bijection). As far as we know, this bijection is new and not equivalent to any of the well-known bijections between planar maps and tree-like objects.

This work is a step towards a unified bijective explanation of recurrence formulas on maps arising from the KP hierarchy. We state here the two formulas that motivate this work. Recall that $\tau(n, g)$ is the number of triangulations of genus g with $2n$ triangles. The Goulden–Jackson formula [GJ08] allows to compute the coefficients $\tau(n, g)$ recursively:

$$\begin{aligned}(n+1)\tau(n,g) =&4n(3n-2)(3n-4)\tau(n-2,g-1)+4(3n-1)\tau(n-1,g)\\ &+4\sum_{\substack{i+j=n-2\\ i,j\geqslant 0}}\sum_{\substack{g_1+g_2=g\\ g_1,g_2\geqslant 0}}(3i+2)(3j+2)\tau(i,g_1)\tau(j,g_2)+2\mathbb{1}_{n=g=1}.\end{aligned} \tag{4.0.1}$$

By duality of maps, $\tau(n, g)$ is also the number of rooted cubic maps (maps with only vertices of degree 3) of genus g with $2n$ vertices. The number $Q_g(n, f)$ of maps of genus g with n edges and f faces can be computed via the Carrell–Chapuy formula [CC15]:

$$(n+1)Q_g(n,f) = 2(2n-1)Q_g(n-1,f) + 2(2n-1)Q_g(n-1,f-1) \\ + (2n-3)(n-1)(2n-1)Q_{g-1}(n-2,f) \\ + 3\sum_{\substack{i+j=n-2\\ i,j\geqslant 0}}\ \sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}\ \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geqslant 0}} (2i+1)(2j+1)Q_{g_1}(i,f_1)Q_{g_2}(j,f_2), \tag{4.0.2}$$

Taking $f = 1$ in (4.0.2), one recovers the famous Harer–Zagier recurrence [HZ86]. This formula was proven bijectively in [CFF13], and it was the first bijective work on a formula linked to the KP hierarchy (and the only one before our work, as far as we know). Here, we present bijective proofs for the planar case ($g = 0$) of the Goulden–Jackson and Carrell–Chapuy formulas (4.0.1) and (4.0.2). Note that contrary to the one-faced case, which is linear, the planar formulas are, as in the general case, quadratic.

The Carrell–Chapuy formula (4.0.2) comes from two separate (but somehow related by their bijective proofs) formulas that were not predicted by the KP hierarchy. One of them is the consequence of a generalization of the famous Rémy bijection on plane trees [Rém85] to all planar maps (see formulas (4.1.1) and (4.1.3)). One of the bijections can be refined to control the distribution of vertex degrees. As a consequence, we derive a general formula on precubic maps (see Section 4.1 for a definition) which implies the Goulden–Jackson formula (4.0.1).

Our bijections rely on a particular exploration of the map and on a "cut-and-slide" operation. Although non-local, this operation allows us to keep track of the degrees of the vertices (see Section 4.4, Theorem 4.4.1). We also take a first step towards the higher genus case by proving the Goulden–Jackson formula for two-faced maps in any genus (see Section 4.6 for a sketched proof).

Discussion and related works. The bijective study of planar maps is a well understood topic, especially thanks to bijections with tree-like objects. However, as far as we know, our bijection is not equivalent to those bijections (although certain similarities can be observed, such as a search of the dual map, depth first search in our case, breadth first search in the bijections with tree-like objects). A related cut-and-slide operation appeared in [Bet14], but it is different from ours (see Remark 4.2.6 for more details).

It is a natural question to try to unify bijections presented in this chapter (that apply to planar maps with arbitrary number of faces) and bijections in [CFF13] (that apply to unicellular maps with arbitrary genus) in order to prove (4.0.1) and (4.0.2) in full generality. These two approaches seem different at first sight, but they both consist of taking a map with a marked special half-edge, and "cutting" at this special half-edge in order to obtain one or two maps with fewer edges, faces, or lower genus. In our case, those special half-edges are called *discoveries*, and there are $f-1$ of them in a map with f faces; in [CFF13], they are called *trisections*, and there are $2g$ of them in a map of genus g. In (4.0.1), by the Euler formula, the factor $(n+1)$ in the RHS is equal to $2g+f-1$ if there are f faces, this suggests that in the general case these special half-edges still exist and there would be $2g+f-1$ of them. They would thus be a common generalization of trisections and discoveries. We already found a bijective proof of the precubic recurrence formula for two faced maps (a sketch of proof is included in Section 4.6), but it is already quite complicated and involves several cases. That is why putting it all together seems to be a challenge of its own. In another direction, we can also look at the second KP equation (which is also quadratic), which yields a recurrence formula (very similar to (4.1.2)) for planar maps with vertices of degrees 1 or 4, and this formula is also proven by our bijection (it is a specialization of the more general formula obtained in Section 4.4). This suggests that this exploration+cut-and-slide scheme is maybe somehow underlying in the whole KP hierarchy for maps. Finally, the same bijective question applies to the other formulas for maps arising from the KP and 2-Toda hierarchies, namely the formulas of [KZ15] and Chapter 5.

Structure of the chapter. In Section 4.1, we will give some definitions on maps and state the main results. The bijections will be described in Section 4.2 (including the proof of the Goulden–Jackson formula), with some proofs postponed to Section 4.3. Section 4.4 will present refined formulas with control over the degrees of the vertices. In Section 4.5, we prove the Carrell–Chapuy formula using formulas (4.1.1) and (4.1.3). In Section 4.6, we give a sketch of proof of the precubic recurrences for two-faced maps in any genus.

4.1 Definitions and main results

We briefly recall some definitions on maps:

Definition 4.1.1. A map M is the data of a connected multigraph (multiple edges and loops are allowed) G (called the underlying graph) embedded in a compact connected oriented surface S, such that $S \setminus G$ is homeomorphic to a collection of disks. The connected components of $S \setminus G$ are called the *faces*.

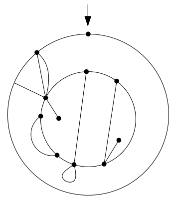


Figure 4.1 – A planar map.

M is defined up to homeomorphism. Equivalently, M is the data of G and a *rotation system* which describes the clockwise cyclic order of the half-edges around each vertex. The *genus* g of M is the genus of S (the number of "handles" in S). A *corner* of M is an angular sector between two consecutive half-edges around a vertex. A *rooted map* is a map with a distinguished corner. A small arrow is placed in the distinguished corner, thus splitting the corner in two separate corners (left and right of the arrow). If a rooted map M of genus g has n edges, v vertices and f faces, the Euler formula links those quantities: $v - n + f = 2 - 2g$. The map M has $2n + 1$ corners.

A *planar map* (see Figure 4.1) is a rooted map of genus 0. It can be drawn on the plane with the root lying on the outer face. A *precubic map* is a map with vertices of degree 1 or 3 only, rooted on a vertex of degree 1. A *leaf* is a vertex of degree 1 that is not the root.

From now on, all the maps we consider will be rooted, and planar unless stated otherwise.

In Section 4.2, we will introduce the concept of *discovery*, which are special edges obtained by an exploration of the map. An important feature is that a map with f faces has $f - 1$ discoveries.

We are now ready to state the two main theorems of this chapter.

Theorem 4.1.1. *There is a bijection between planar maps M with a marked discovery and pairs of planar maps (M_1, M_2) such that M_1 has a marked vertex and M_2 has a marked leaf. This bijection preserves the total number of edges and faces. This gives the following formula on planar maps*

$$(f-1)Q(n,f) = \sum_{\substack{i+j=n-1 \\ i,j \geqslant 0}} \sum_{\substack{f_1+f_2=f \\ f_1,f_2 \geqslant 1}} v_1 Q(i,f_1)(2j+1)Q(j,f_2), \tag{4.1.1}$$

where $Q(n,f)$ is the number of rooted planar maps with n edges and f faces, and v_1 counts the number of vertices in the first map (i.e. $v_1 = 2 + i - f_1$).

This bijection adapts to precubic maps, and the marked vertex of M_1 is now a marked leaf (see Remark 4.2.4). This yields

$$(f-1)\alpha(n,f) = \sum_{i+j=n} \sum_{f_1+f_2=f} \alpha^{(1)}(i,f_1)\alpha^{(1)}(j,f_2), \tag{4.1.2}$$

where $\alpha(n, f)$ counts the number of (planar) precubic maps with n edges and f faces, and $\alpha^{(1)}(n, f)$ counts the number of precubic maps with n edges and f faces and a marked leaf.

Retracting a leaf into a corner (see Figure 4.3 left) is a classical operation that gives a bijection between planar maps M with a marked leaf and planar maps M' with a marked corner, such that M' has as many faces and one edge less than M. This operation, along with a more subtle retraction operation on internal nodes, is at the core of the Rémy bijection on plane trees [Rém85]. We generalize the operation on internal nodes to all planar maps in the following:

Theorem 4.1.2 (Generalized Rémy bijection)**.** *There is a bijection between planar maps M with a marked node (i.e. a vertex that is not a leaf) on the one hand, and the union of planar maps M' with a marked corner and pairs of planar maps (M_1, M_2) such that they both have a marked vertex on the other hand. The total number of faces is preserved, and the total number of edges decreases by one.*

By this bijection, there are

$$(2n-1)Q(n-1, f) + \sum_{\substack{i+j=n-1 \\ i,j \geqslant 0}} \sum_{\substack{f_1+f_2=f \\ f_1,f_2 \geqslant 1}} v_1 Q(i, f_1) v_2 Q(j, f_2)$$

planar maps with n edges, f faces and a marked node. The retraction operation (of Figure 4.3) implies that there are $(2n-1)Q(n-1, f)$ planar maps with n edges, f faces and a marked leaf. This yields the following equation:

$$vQ(n, f) = 2(2n-1)Q(n-1, f) + \sum_{\substack{i+j=n-1 \\ i,j \geqslant 0}} \sum_{\substack{f_1+f_2=f \\ f_1,f_2 \geqslant 1}} v_1 Q(i, f_1) v_2 Q(j, f_2), \quad (4.1.3)$$

where the "v-variables" count the number of vertices (i.e. $v = 2 + n - f$, $v_1 = 2 + i - f_1$ and $v_2 = 2 + j - f_2$). This holds for $n > 0$.

Taking $n = 3m + 2$ and $f = m + 2$ in (4.1.2), one recovers:

Corollary 4.1.3 (Goulden–Jackson (4.0.1) planar case)**.** *We have*

$$(n+1)T(n) = 4(3n-1)T(n-1) + 4 \sum_{\substack{i+j=n-2 \\ i,j \geqslant 0}} (3i+2)(3j+2)T(i)T(j),$$

where $T(n)$ counts the number of planar cubic maps with $3n$ edges.

Remark 4.1.4. *The term $4(3n-1)T(n-1)$ corresponds to $i = f_1 = 1$ and $j = f_2 = 1$ in the summation of (4.1.2) (because $\alpha^{(1)}(1, 1) = 1$).*

Combining (4.1.1) and (4.1.3) and doing some manipulations, one recovers

Corollary 4.1.5 (Carrell–Chapuy (4.0.2) planar case)**.**

$$\begin{aligned}(n+1)Q(n,f) =& 2(2n-1)Q(n-1,f) + 2(2n-1)Q(n-1,f-1)\\ &+ 3\sum_{\substack{i+j=n-2\\ i,j\geqslant 0}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(2i+1)(2j+1)Q(i,f_1)Q(j,f_2).\end{aligned} \quad (4.1.4)$$

In Section 4.2, we will give the definition of the exploration and the discoveries, justify (4.1.1) and (4.1.2) and Corollary 4.1.3, then we will describe the bijections behind Theorems 4.1.1 and 4.1.2. The proof that these are indeed bijections can be found in Section 4.3. Formula (4.1.4) is not straightforwardly derived from (4.1.1) and (4.1.3). The calculations to recover (4.1.4) from there are displayed in Section 4.5.

4.2 The bijections

This section is organized as follows: first, we define the exploration process of our map. Then, after introducing some key notions like the discoveries, we present the cut-and-slide bijection of Theorem 4.1.1. Finally, we present the generalized Rémy bijection of Theorem 4.1.2.

Some of the important lemmas are stated here, but the proof that the bijections we present are indeed bijections will be in Section 4.3.

4.2.1 The exploration

Definition 4.2.1. The *exploration* of a planar map (see Figure 4.2) is defined iteratively in the following way: starting from the root, go along the edges, keeping the edges on the right (progress in clockwise order). When an edge that is at the interface of the current face and a face not yet discovered is found, open this edge into a bud (an outgoing half-edge) and a stem (an ingoing half-edge). The rule is that the bud has to appear before the stem in the exploration (see Figure 4.2). Continue the process, thus entering the new face. Continue until the root is reached again.

Each edge that has been opened during the process is called a *discovery*, and the vertex attached to the bud is called a *discovery vertex*. If there are f faces , there are $f-1$ discoveries (note that several discoveries can share the same discovery vertex).

The exploration is actually equivalent to a depth first search of the dual, with a "right first" priority. Thus for each face but the outer face we can define its *previous face* as the face that is its parent in the spanning tree of the dual found by the exploration. The notion of *previous discovery* can be similarly defined. This also defines an order on the corners (resp. half-edges) incident to each vertex, according to the order in which they were visited during the exploration (see Figure 4.2).

Let e be a discovery, incident to faces f_1 and f_2, such that f_1 is the previous face of f_2. We say e *leaves* f_1 and *enters* f_2.

Remark 4.2.1. *The exploration is a dynamic process that modifies the map along the way, but in the end, once the exploration is over and the discoveries have been found, we will deal with the original, unmodified map, with its original edges and faces. It is as if we did the exploration then closed the map back. Alternatively, one can think of an exploration that does not open the discoveries but just crosses them.*

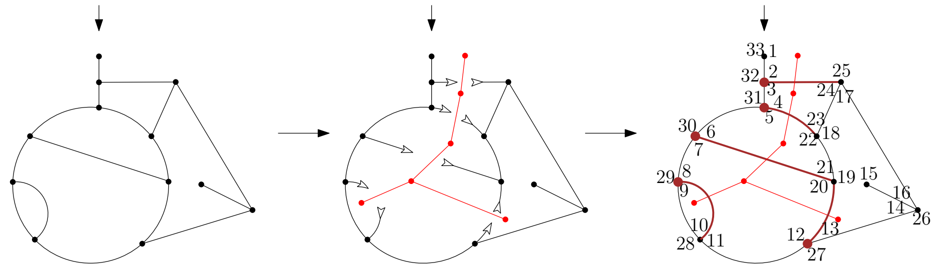


Figure 4.2 – The exploration of a planar map. The buds are the outgoing arrows, the stems are the ingoing arrows. Left: the original map. Center: the opened map. Right: The original map, with its discoveries and discovery vertices in fat brown. The red tree describes the partial order among the faces. The corners are labelled in the order they were found during the discovery.

Lemma 4.2.2. *Around each vertex, the order of the corners as defined by the exploration agrees with the clockwise order.*

Proof. The order between the corners of the map is exactly the same as the order of the corners of the blossoming tree (i.e. a tree with some buds and stems attached) obtained by opening the discoveries. The exploration of a tree is just a tour of the unique face, and it is clear that the corners of each vertex are in clockwise order, and we are done. □

We can now explain how the formulas are a consequence of the bijections. In a map with f faces, there are $f-1$ discoveries, so there are $(f-1)Q(n,f)$ maps with n edges, f faces and a marked discovery. A marked leaf can be retracted into a marked corner (see Figure 4.3 left), such that there are $(2j+1)Q(j, f_2)$ maps with $j+1$ edges, f_2 faces and a marked leaf. There are $v_1 Q(i, f_1)$ maps with i edges, f_1 faces and a marked vertex. This explains why, in Theorems 4.1.1 and 4.1.2, Formulas (4.1.1) and (4.1.3) are indeed consequences of the bijections. In a precubic map, one can retract a leaf into a marked side-edge losing two edges: after retracting the leaf into a corner, merge the two edges adjacent to it (see Figure 4.3 right). The same operation can be performed on the root of a precubic map, thus a precubic map with $3n+2$ edges and no leaves is equivalent to a cubic map with $3n$ edges. This explains how to derive Corollary 4.1.3 from (4.1.2).

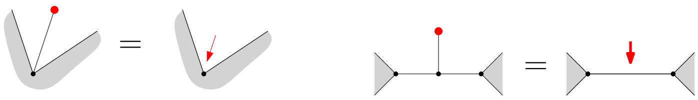


Figure 4.3 – Retracting a leaf: in a general map (left), in a precubic map (right, the vertex adjacent to the leaf is also removed).

4.2.2 Cut-and-slide bijection

We will describe the bijection of Theorem 4.1.1 between maps M with a marked discovery and pairs of planar maps (M_1, M_2) such that M_1 has a marked vertex and M_2 has a marked leaf. It will then be straightforward to see that restricting the bijection to precubic maps gives (4.1.2).

We first need to introduce the notion of disconnecting discovery, and the splitting operation.

Definition 4.2.2. A discovery is said to be *disconnecting* if the corner preceding the discovery and the last corner (in the order defined by the exploration) around the discovery vertex lie in the same face.

Splitting operation. *Any map with a marked disconnecting discovery can be split into two maps, one with a marked vertex, the other with a marked leaf in the outer face (see Figure 4.4): split the discovery vertex in two between* c *and* c^*. *This divides the map in two, one containing the original root and a marked vertex, the other one rooted on the "splitting corner". Detach the discovery from the root to obtain a marked leaf in the outer face. This operation is bijective: to go back, reattach the marked leaf to the root of the second map, and then glue its root to the last corner around the marked vertex in the first map.*

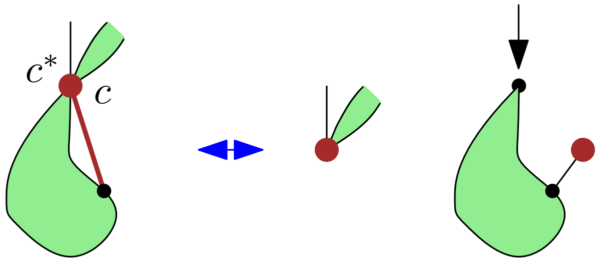


Figure 4.4 – Splitting a map at a disconnecting discovery. Here we only see what happens locally around the disconnecting discovery. On the left, the discovery and its discovery vertex are in fat brown, c is the corner preceding the discovery, and c^* is the last corner around the discovery vertex.

The splitting operation of a disconnecting discovery describes our bijection in the case where the marked discovery is disconnecting, and the reverse bijection in the case where the marked leaf lies in the outer face.

In general, discoveries are not disconnecting. In order to still split the map in two, given a discovery e, we will need to find a disconnecting discovery e' that is "canonically" related to e. Conversely, the marked leaf in M_2 is not always in the outer face, we will need to "propagate" the leaf all the way up to the outer face. The notion of previous discovery will help us with that. This will be the general construction of the bijection that is detailed below. First we need to ensure that there will always be a disconnecting discovery. That is the purpose of Lemma 4.2.3.

Lemma 4.2.3. *If a discovery leaves*[1] *the outer face, then it is disconnecting.*

Proof. We will show the following stronger result, which directly implies Lemma 4.2.3: if a vertex has a corner in the outer face, then its last corner lies in the outer face. Indeed, if a discovery leaves the outer face, then the corner preceding the discovery must be on the outer face. In particular, if v denotes the discovery vertex, then v has a corner on the outer face, so by this result the last corner of v must be on the outer face. Thus, v is a disconnecting discovery.

In a map M, let v be a vertex with one of its corners lying in the outer face. If v is the root vertex, then it is obvious that its last corner lies in the outer face. Else, let e be the first edge around v, as defined by the exploration (i.e. the edge we see in the exploration just before we see v for the first time). Let c be a corner around v that lies in the outer face. Suppose it is not the

[1]as defined in Definition 4.2.1.

last corner around v (if it is, we have nothing to prove). Let c_0 (resp. c^*) be the first (resp. the last) corner around v, and let F_0 (resp. F^*) be the face in which c_0 (resp. c^*) lies (see Figure 4.5).

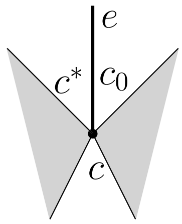


Figure 4.5

If a face F' is deeper (in terms of the partial order defined by the exploration) than a face F (and $F \neq F'$), then during the exploration (ignoring all other faces), we first see a part of F, then all of F', then the rest of F. This implies that, if F^* is not the root face, during the exploration of M, at the time we see c, F^* has not been discovered yet. But that would mean that at the time we see e (which comes before c in the exploration), F^* has not been discovered yet. But then e would be a discovery, and thus c^* would be seen before c_0, that is a contradiction. The lemma is proved. □

Bijection 1. *The general process is iterative (see Figure 4.6 for an example).*

Cut process: *Start from a map M with a marked discovery e, let v be its discovery vertex. If the discovery is disconnecting, then split M at v as described in the splitting operation.*

Otherwise, open e into a bud b_0 and a stem s_0, and consider its previous discovery e_1 (in the order defined above). If it is disconnecting, then split it, otherwise open it (into b_1 and s_1) and consider the previous discovery e_2, and so on until a splitting operation is made. Note that a discovery that leaves the outer face is always disconnecting (because of Lemma 4.2.3), so the algorithm terminates. One ends up with two maps M_1 and M_2', such that M_1 has a marked vertex and M_2' has a marked leaf l and (possibly) some buds and stems, all lying in the outer face.

Slide process: *We will not modify M_1. If there are no buds and stems in M_2', we are done. Else, consider s_0, and make it a marked leaf. Then consider l, and make it a stem. Finally, glue back the buds and stems together canonically: starting from the root of M_2', taking a clockwise tour of the outer face, one encounters a certain number of buds, then the same number of stems. There is only one way to match each bud with each stem such that the*

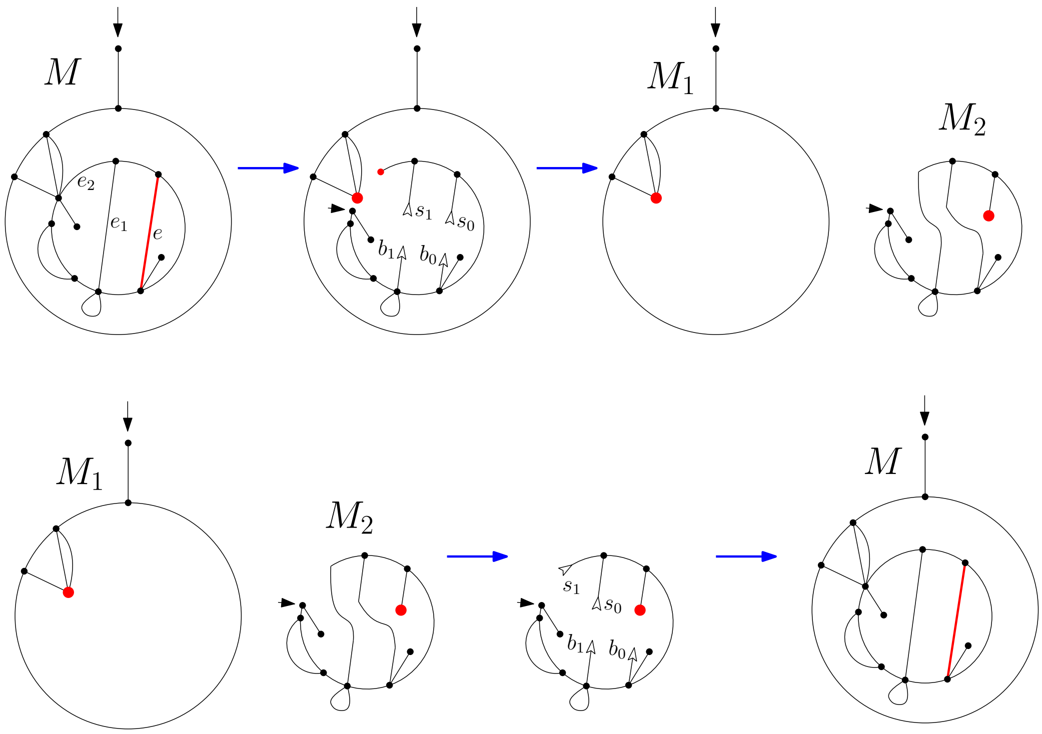


Figure 4.6 – The bijection (above) and its inverse (below).

map remains planar. Equivalently, if there are $k+1$ buds and $k+1$ stems, match b_0 with s_1, and so on, until b_k is matched with l. We obtain a map M_2 with a marked leaf, together with the map M_1 with a marked vertex.

We can now describe the inverse bijection:

Bijection 2. *Starting from M_2 with a marked leaf l and M_1 with a marked vertex, consider M_2. The leaf l lies in a certain face F, and if F is not the outer face, there is a certain discovery e_0 that enters F. Open it into a bud b_0 and a stem s_0, then open the previous discovery e_1, and repeat the process until a discovery that leaves the outer face has been opened (in that case there is no previous discovery to open). One ends up with a map M_2' with a marked leaf l and possibly some buds and stems, all lying in the outer face. If there are some buds and stems, let s be the stem that was created last in the process. Make s a marked leaf l^*, and make l a marked stem s^*, then close the map canonically. One now has a map M_2^* with a marked leaf on the outer face and (possibly) a marked edge e (that comes from the closure of s^*). This marked edge is actually a discovery in M_2^* (and will be a discovery in the final map). If e does not exist, let $l^* = l$, and mark the edge adjacent to l^* (and call it e). Now do the inverse of the splitting operation: glue l^* to the* *root vertex of M_2^* at its first corner, and then glue the root of the resulting map at the last corner of the marked vertex of M_1 to obtain a map M with a marked discovery e.*

Remark 4.2.4. *This operation restricts to precubic maps, and in this case discovery vertices are always of degree* 3*, so that when split, the marked vertex of M_1 and the root of M_2 are both of degree* 1*. This justifies* (4.1.2)*.*

Those two operations are inverse of each other because of the following property (that will be proved in Section 4.3):

Lemma 4.2.5. *In the bijection and its inverse, the closure of the buds and stems are discoveries in the resulting map. Moreover, in the bijection, if $e_1, \dots e_k$ are the discoveries created by closing buds and stems, e_k leaves the outer face, and for all $i < k$, e_{i+1} is the previous discovery of e_i. In the inverse bijection, if $e_1, \dots e_{k+1}$ are the discoveries created by closing buds and stems, e_{k+1} is disconnecting, and for all $i < k$, e_{i+1} is the previous discovery of e_i.*

Remark 4.2.6. *A similar but different cut-and-slide operation also appeared in [Bet14] (and in some sense also in [AFP14]). However there are significant differences: in [Bet14] the cut path is geodesic (leftmost BFS), and both endpoints of the path need to be specified. Whereas here, the cut path is defined by a leftmost DFS, and only one endpoint of the path (the marked discovery) needs to be specified, the other endpoint (the disconnecting discovery) is uniquely determined. Furthermore, the bijections in [Bet14] imply linear formulas, contrary to our quadratic formulas.*

4.2.3 Generalized Rémy bijection

We will now describe a generalized Rémy bijection (Theorem 4.1.2) on planar maps, that also relies on the cut-and-slide operation.

We recall Rémy's bijection for plane trees that proves the formula

$$(n+1)Cat(n) = 2(2n-1)Cat(n-1)$$

where $Cat(n)$ counts the number of rooted plane trees with n edges (see Figure 4.7 for an example). Start from a tree T with n edges and a marked vertex v. If v is a leaf, retract it (as in Figure 4.3) to obtain a tree T' with $n-1$ edges, with a marked corner. Otherwise, v has a last child v', that is its child that is found last during a clockwise tour of the unique face. We can then contract the edge between v and v', and mark a corner around the merging vertex to remember where to grow the edge back (as in Figure 4.7). The reverse bijection is then straightforward.

Remark 4.2.7. *Leaves are defined as vertices of degree* 1 ***that are not the root vertex****. Thus, if the root vertex has degree* 1*, one should apply the second operation to it.*

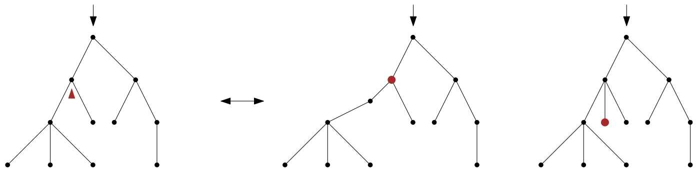


Figure 4.7 – Rémy's bijection for plane trees. The left tree can be obtained by applying the bijection to each of the two trees on the right. The rightmost tree corresponds to the case where v is a leaf, and the middle tree corresponds to the case where v is not a leaf.

In a general planar map, let us define precisely the operations of edge contraction and edge growing (See Figure 4.8):

Let v be a non-leaf vertex. It has a last corner c_2 (as defined by the exploration). Let e be the edge that comes just before c_2 in clockwise order around v. We call e the last edge around v. Let v' be the other end of e (note that it is possible that $v = v'$). It is called the last child of v. We will define the contraction operation only in the case where v is seen strictly before v' in the exploration. The edge e is adjacent to two corners of v: c_2 and another corner c_1. It is also adjacent to two corners of v', c' and c''. Since v appears strictly before v', c' is the first corner around v', and c'' is the last corner around v'.

The contraction operation consists in contracting e, merging v and v' into a vertex v^*, merging c_1 and c' into c^* and c_2 and c'' into c, such that c is the last corner around v^*, and c^* is the marked corner.

Conversely, starting with a marked corner c^* adjacent to a vertex v^* whose last corner is c, one can grow an edge e, splitting v^* into v and v', c^* into c_1 and c', and c into c^* and c_2. The corner c_2 is the last corner around v and c'' the last corner around v'. The marked vertex is v, and v' is its last child. This defines the growing operation. Lemma 4.2.8 will ensure these two operations are inverse of each other.

Lemma 4.2.8. *Let M be a map with a marked corner c^*, and let M' be the map obtained after a growing operation is performed at c^*. Let v be the marked vertex and v' its last child. Then v appears before v' in the exploration.*

Proof. The growing operation is local, so both explorations of M and M' look exactly the same before c^* (resp. c_1) is found. Let p be the first corner

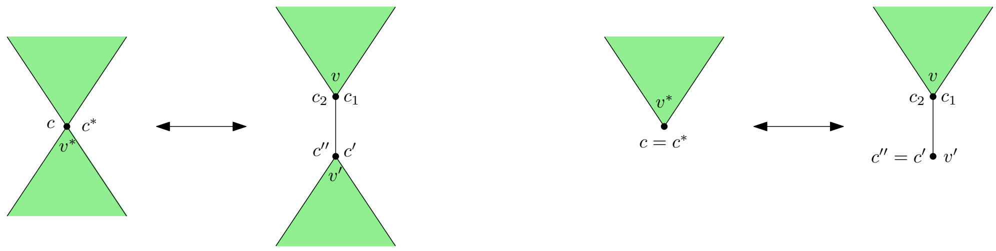


Figure 4.8 – The contraction and growing operations (right: the special case where the marked corner is the last corner).

around v^* in M. After the growing operation it is adjacent to v, and it is reached before any other corner around v or v', thus v appears before v' in the exploration (the special case $p = c^*$ works the same). $\square$

We also need to cover the remaining case, namely when the marked vertex is seen after its last child in the exploration.

Lemma 4.2.9. *Let v be a non-leaf vertex, e its last edge, and v' its last child. If v' is found before v in the exploration (including the case $v = v'$), then e is a discovery.*

Proof. If $v = v'$, then e is a loop, thus a discovery (since a loop separates a planar map in two, it has to be opened during the exploration).

Otherwise, let c_1, c_2 (resp. c', c'') be the corners of v (resp. v') that are adjacent to e. Note that c_2 is the last corner of v, and assume e is not a discovery. Thus, it is not opened during the exploration, such that c'' comes before c_2, and c_1 before c' (see Figure 4.9). The corner c' cannot be the first corner of v', otherwise v would be seen before v' in the exploration. Thus, Lemma 4.2.2 implies that c' comes after c''. But that would mean the order among the four corners in the exploration is c'', then c_2, then c_1, then c', which is impossible since c_2 is the last corner of v. Thus, e is a discovery. $\square$

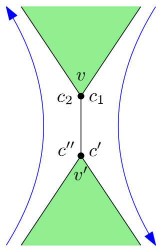


Figure 4.9 – The exploration locally around e, if it is not a discovery.

Now we are ready to explain the generalized Rémy bijection (see Figure 4.10):

Bijection 3. *Take a planar map M and mark a vertex v. If it is a leaf, contract it to mark a corner. Else, let v' be its last child, and e be its last edge. If v' is seen after v in the exploration, then contract e and mark the corresponding corner. The inverse operation is the growing operation as described above. This gives the first term of the RHS of* (4.1.3)*. Otherwise, e is a discovery, apply the cut-and-slide operation at e. One ends up with two maps (M_1, M_2). M_1 has a marked vertex, and M_2 has a marked leaf l that is the last child of its neighbour w. Contract l to mark w. To go backwards, grow l out of the last corner around w, then apply the inverse of the cut-and-slide operation. This gives the second term of the RHS of* (4.1.3).

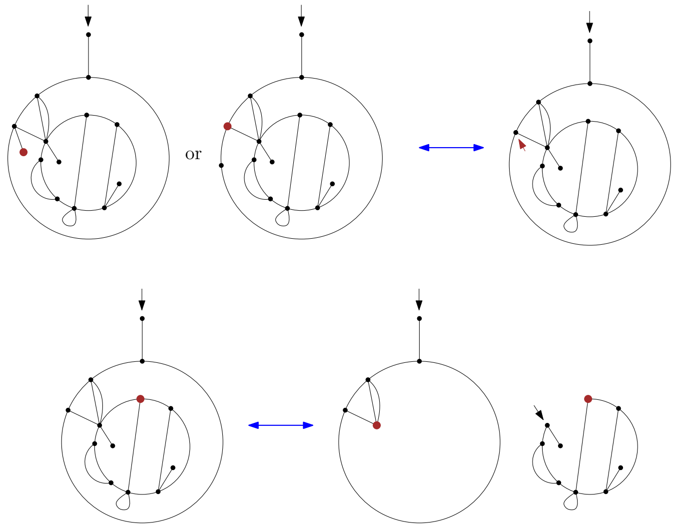


Figure 4.10 – The generalized Rémy bijection. The case similar to trees (above), and the cut-and-slide case (below).

4.3 Proofs

In this section we will present a proof that Bijections 1 and 2 are inverse to each other, thus implying that the cut-and-slide operation is a bijection. Then, together with the lemmas presented in Section 4.2.3, it will be enough to prove that the generalized Rémy bijection (Bijection 3) is indeed a bijection. To do so, we will describe what happens to the different faces of the map during the operation (see Figure 4.11).

Proof of Lemma 4.2.5. We only prove Lemma 4.2.5 for Bijection 1. The proof for Bijection 2 is very similar and can be copied almost verbatim.

Thanks to the exploration, we can consider faces as words on the alphabet of side-edges (each face is described by the list of the side-edges that lie inside it, ordered as they appear during the exploration). For any side-edge e, we will call $\overline{e}$ its opposite side-edge (i.e the other side of this edge). By abuse of notation we may also refer to the edge e that is the edge made of the side-edges e and $\overline{e}$, but we will stress the fact that we are referring to an edge.

Fix a planar map M. Let d_0 be a discovery, d_1 its previous discovery (if it exists), and so on, d_{i+1} being the previous discovery of d_i, as long as it exists. Let (F_i) be the sequence of faces of M such that the discovery d_i discovers the face F_i while leaving face F_{i+1}. For all i, let v_i be the discovery vertex corresponding to d_i, and let us tell apart both side-edges of the discovery: d_i sits in F_{i+1} and $\overline{d_i}$ sits in F_i. Let k be the smallest i such that d_i is a disconnecting discovery.

Now, we will describe faces as words of side-edges, in the order they are visited by the exploration. For all i between 1 and k, we can say that $F_i = \overline{d_i}A_i d_{i-1} B_i$, i.e. A_i are the side-edges of F_i encountered before d_{i-1} in the exploration, and B_i are the the side-edges of F_i encountered after d_{i-1} in the exploration. We also set $F_0 = \overline{d_0}A_0$ and $F_{k+1} = \overline{d_{k+1}}C d_k B_{k+1} C'$ where B_{k+1} is the word of side-edges of F_{k+1} that appear between d_k and the last corner around v_k.

Now let us describe the cut-and-slide operation when d_0 is marked. First, for all $i < k$, all **edges** d_i are opened into a bud b_i and a stem s_i. The **edge** d_k becomes a marked leaf s_k. Then s_k becomes a stem, and s_0 becomes the marked leaf. Then b_i is matched with s_{i+1} to create an edge e_i. The map has been split into two distinct maps M_1 and M_2, now let us see what happened to the faces. In M_1, there is only one face that has been modified, it is the face in which lies the last corner of the marked vertex v^*, that we call G^*. It is straightforward to see that $G^* = \overline{d_{k+1}}CC'$ where the last corner of v^* appears exactly between the end of C and the beginning of C'. In M_2, the modified faces will be called G_i. We have $G_0 = \overline{e_0}A_0 s_0 \overline{s_0} B_1$, $G_i = \overline{e_i}A_i e_{i-1} B_{i+1}$ and $G_k = A_k e_{k-1} B_{k+1}$. The outer face of M_2 is G_k. We must now prove that, for all $i < k$, the **edge** e_i is a discovery that leaves G_{i+1} and discovers G_i. We already know that e_i lies in G_{i+1} and $\overline{e_i}$ lies in G_i, we just have to prove that the **edge** e_i is a discovery, i.e. e_i is the first side-edge e of G_{i+1} such that $\overline{e}$ lies in G_i. Assume the contrary. There must be a side-edge f in A_{i+1} such that $\overline{f}$ lies in G_i. There are two possibilities:

1. Either $\overline{f}$ belongs to A_i, but then in the original map, in F_{i+1}, f comes

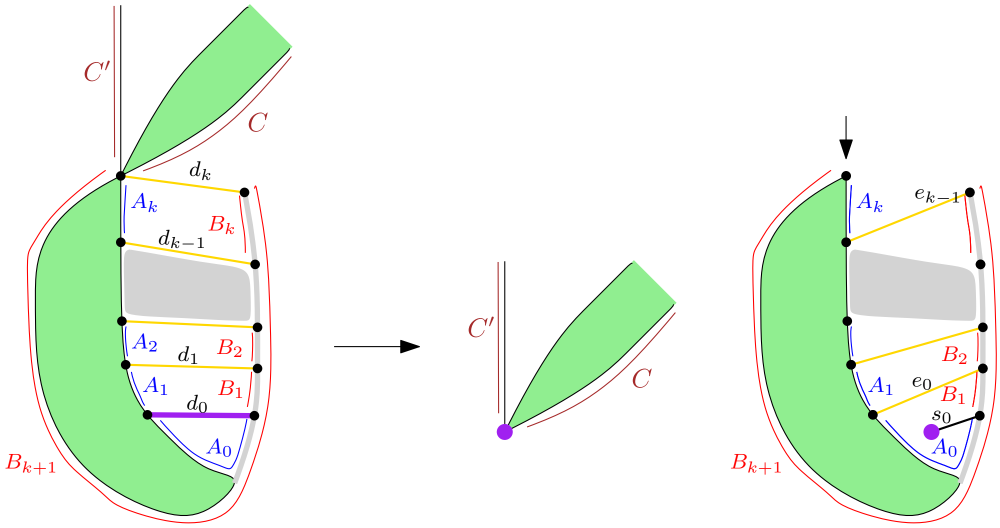


Figure 4.11 – What happens to the faces during the cut-and-slide operation. Marked objects are in fat purple, other discoveries involved are in gold. Blue contour materializes the A_i's and red contour the B_i's.

before d_i and $\overline{f}$ lies in F_i, thus d_i could not be the discovery, in this case there would be a contradiction.

2. Or $\overline{f}$ belongs to B_{i+1}, but then in the original map, in F_{i+1} we see the side-edges f, then d_i, then $\overline{f}$ in that order. But since we deal with planar maps, an edge which has both sides in the same face is a cut-edge (removing it disconnects the map). But then in that configuration, d_i must be a disconnecting discovery (see Figure 4.12). Indeed, let M^* be the connected component containing d_i that we obtain after removing e. Then, in the map M^*, d_i would be a discovery that leaves the outer face, thus disconnecting (because of Lemma 4.2.3). This is a contradiction, because $i < k$.

We proved what we wanted.

□

Knowing exactly what happens to the faces and that the discoveries are in some sense "preserved" is enough to prove that both the cut-and-slide operation and its inverse work fine and are actually inverse operations of each other.

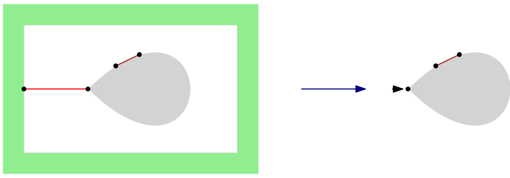


Figure 4.12 – The impossibility of a cut-edge separating a non-disconnecting discovery from the root. In red, the cut-edge, in brown the discovery.

4.4 Controling the degrees of the vertices

Here we present an analogue of (4.1.1) with control over the degrees of the vertices. This is a result of independent interest that is not used in the other proofs of the chapter.

If $\mathbf{v} = (v_i)_{i\in\mathbb{N}}$ is a sequence of integers, for any $j > 0$, we set $\delta_j(\mathbf{v}) = \mathbf{w}$ where $w_j = v_j + 1$ and $w_i = v_i$ for $i \neq j$, and $\delta_{-j}(\mathbf{v}) = \mathbf{w}'$ where $w'_j = v_j - 1$ and $w'_i = v_i$ for $i \neq j$. Finally, we set $\delta(\mathbf{v}, j_1, \dots, j_k) = \delta_{j_1} \circ \dots \circ \delta_{j_k}(\mathbf{v})$.

Let $M(r, f, \mathbf{v})$ be the number of planar maps with f faces, with root of degree r, with $\mathbf{v} = (v_i)_{i\in\mathbb{N}}$ such that there are v_i vertices of degree i (root included). The cut-and-slide operation only modifies the degrees at the marked leaf and the splitting vertex, so we can immediately derive this more precise formula:

Theorem 4.4.1.

$$
\begin{aligned}
(f-1)M(r,f,\mathbf{v}) = &\sum_{j,k\,\geqslant\,1}\sum_{\mathbf{u},\mathbf{w}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\,\geqslant\,1}} (u_j - \mathbb{1}_{j=r})M(r,f_1,\mathbf{u})(w_1-\mathbb{1}_{k=1})M(k,f_2,\mathbf{w})\\
&+\sum_{j+k=r-1}\sum_{\mathbf{u},\mathbf{w}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\,\geqslant\,1}} M(j,f_1,\mathbf{u})(w_1-\mathbb{1}_{k=1})M(k,f_2,\mathbf{w}).
\end{aligned}
$$

where the sum of the first line is over pairs of sequences of numbers $(\mathbf{u}, \mathbf{w})$ *such that* $\mathbf{u} + \mathbf{w} = \delta(\mathbf{v}, 1, j, k, -(j + k + 1))$*, and the sum of the second line over pairs of sequences of numbers* $(\mathbf{u}, \mathbf{w})$ *such that* $\mathbf{u} + \mathbf{w} = \delta(\mathbf{v}, 1, -r, k, j)$*.*

Proof. Notice that, in the cut-and-slide operation, very few of the vertex degrees are modified: there is the leaf that is created in M_2, and the disconnecting discovery vertex that is split in three. Other than that, for all other vertices, although some of their adjacent edges might be split, they are reattached somewhere else so their degree does not change.

Let v be the vertex that is split in three: it gives birth to v_1, the marked vertex in M_1, v_2, the root vertex of M_2, and v_3, that after some possible transformation "becomes" the marked leaf in M_2. Say v is of degree $j+k+1$, with $deg(v_1) = j$ and $deg(v_2) = k$.

The $(w_1 - \mathbb{1}_{k=1})$ term means that the marked leaf in M_2 cannot be the root. Finally, there are two possible cases, depending on whether v is the root of M or not, each implying one term in the RHS. $\square$

Remark 4.4.2. *Writing $\delta_j(\mathbf{v})$ means there is one more vertex of degree j, and $\delta_{-j}(\mathbf{v})$ means there is one less vertex of degree j. This notation is somehow complicated but avoids dealing with special cases, for instance in $\delta(\mathbf{v}, j, k)$ there is no problem if $j = k$, contrary to saying something like $v'_j = v_j + 1$, $v'_k = v_k + 1$ and $v'_i = v_i$ for $i \neq j, k$.*

Note that this recurrence formula allows us to compute the number of maps with bounded vertex degrees, i.e. it can be specialized to maps with vertex degrees less or equal to d for some d. Restricting to vertex degrees 1 and 3, one recovers (4.1.2).

Formula (4.1.3) also has an analog where the degrees are recorded, but it involves more complicated cases and is less useful for enumeration, we leave it as an exercise to the reader.

4.5 The proof of the Carrell–Chapuy formula in the planar case

We are now ready to prove the Carrell–Chapuy formula in the planar case (4.1.4). We will prove it by induction on n, only starting with the information that $Q(0, f) = \mathbb{1}_{f=1}$.

Applying (4.1.1) to the dual map, we obtain the following formula, which will be helpful for the proof:

Corollary 4.5.1.

$$(v-1)Q(n,f) = \sum_{\substack{i+j=n-1 \\ i,j \geqslant 0}} \sum_{\substack{f_1+f_2=f+1 \\ f_1,f_2 \geqslant 1}} (2i+1)Q(i,f_1)f_2Q(j,f_2) \tag{4.5.1}$$

Starting with (4.1.1), and applying (4.1.3), then in a second time applying (4.1.1) backwards, we obtain

$$\begin{aligned}(f-1)Q(n,f) =&(2n-1)Q(n-1,f-1)+2(2n-1)Q(n-1,f)\\
&+2\sum_{\substack{i+j=n-2\\ i,j\geqslant 0}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(2i+1)Q(i,f_1)(2j+1)Q(j,f_2)\\
&+\sum_{\substack{i+j+k=n-2\\ i,j,k\geqslant 0}}\sum_{\substack{f_1+f_2+f_3=f\\ f_1,f_2,f_3\geqslant 1}}(2i+1)Q(i,f_1)v_2Q(j,f_2)v_3Q(k,f_3)\\
&=(2n-1)Q(n-1,f-1)+2(2n-1)Q(n-1,f)\\
&+2\sum_{\substack{i+j=n-2\\ i,j\geqslant 0}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(2i+1)Q(i,f_1)(2j+1)Q(j,f_2)\\
&+\sum_{\substack{i+j=n-1\\ i,j\geqslant 0}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(f_1-1)Q(i,f_1)v_2Q(j,f_2).\end{aligned}$$

So adding (4.1.3) to this

$$\begin{aligned}(n+1)Q(n,f) =&(2n-1)Q(n-1,f-1)+2(2n-1)Q(n-1,f)\\
&+2\sum_{\substack{i+j=n-2\\ i,j\geqslant 0}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(2i+1)Q(i,f_1)(2j+1)Q(j,f_2)\\
&+\sum_{\substack{i+j=n-1\\ i,j\geqslant 0}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(i+1)Q(i,f_1)v_2Q(j,f_2).\end{aligned}$$

Let

$$S=\sum_{\substack{i+j=n-1\\ i,j\geqslant 0}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(i+1)Q(i,f_1)v_2Q(j,f_2).$$

We want to prove

$$S=(2n-1)Q(n-1,f-1)+\sum_{\substack{i+j=n-2\\ i,j\geqslant 0}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(2i+1)Q(i,f_1)(2j+1)Q(j,f_2).$$

We apply the recursion hypothesis:

$$\begin{aligned}S =&\sum_{\substack{i+j=n-1\\ i,j\geqslant 0}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(2(2i-1)Q(i-1,f_1)+2(2i-1)Q(i-1,f_1-1))v_2Q(j,f_2)\\
&+3\sum_{\substack{i+j+k=n-3\\ k,l\geqslant 0}}\sum_{\substack{f_1+f_2+f_3=f\\ f_1,f_2,f_3\geqslant 1}}(2i+1)Q(i,f_1)(2j+1)Q(j,f_2)v_3Q(k,f_3)\\
&+vQ(n-1,f-1).\end{aligned}$$

But, according to (4.1.1),

$$\sum_{\substack{i+j=n-1\\ i,j\geqslant 0}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(2(2i-1)Q(i-1,f_1-1))v_2Q(j,f_2)=2(f-2)Q(n-1,f-1),$$

and

$$\begin{aligned}&\sum_{\substack{i+j+k=n-3\\ k,l\geqslant 0}}\sum_{\substack{f_1+f_2+f_3=f\\ f_1,f_2,f_3\geqslant 1}}(2i+1)Q(i,f_1)(2j+1)Q(j,f_2)v_3Q(k,f_3)\\ &=\sum_{\substack{i+j=n-2\\ i,j\geqslant 0}}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(2i+1)Q(i,f_1)(f_2-1)Q(j,f_2).\end{aligned}$$

So

$$\begin{aligned}S=&(2(f-2)+v)Q(n-1,f-1)\\ &+\sum_{i+j=n-2}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(2i+1)Q(i,f_1)(2j+1)Q(j,f_2)\\ &+\sum_{i+j=n-2}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(2i+1)Q(i,f_1)f_2Q(j,f_2).\end{aligned}$$

But using (4.5.1), we have

$$\sum_{i+j=n-2}\sum_{\substack{f_1+f_2=f\\ f_1,f_2\geqslant 1}}(2i+1)Q(i,f_1)f_2Q(j,f_2)=(v-1)Q(n-1,f-1),$$

which finishes the proof.

Remark 4.5.2. *The proof above is not straightforward, and since it uses duality, our method cannot be applied to finding an extended Carrell–Chapuy formula with control over the degrees (if it even exists).*

4.6 A bijection for precubic maps with two faces

In this section, we will give a sketch of the proof of a first step towards uniting higher genus and multiple faces: the case of two-faced precubic maps.

Similarly as (4.1.2), we have a recurrence formula for higher genus precubic maps with two faces:

$$(2g+1)\alpha_g(n,2)=\alpha^{(3)}_{g-1}(n,2)+\sum_{g_1+g_2=g}\sum_{i+j=n}\alpha^{(1)}_{g_1}(i,1)\alpha^{(1)}_{g_2}(j,1)\qquad(4.6.1)$$

where $\alpha_g(n, f)$ counts the number of precubic maps with n edges and f faces, of genus g, and $\alpha_g^{(k)}(n, f)$ counts the number of those maps with k marked leaves.

We can define the exploration of a precubic map with two faces in the following way:

Definition 4.6.1. We will describe an exploration of the map as a canonical labelling of all its corners (see Figure 4.13).

As in Definition 4.2.1, we can define the discovery as the first edge adjacent to both faces that is encountered in a clockwise tour of the root face starting from the root, and the discovery vertex as the vertex that appears right before the discovery in this tour (there is only one discovery since there are 2 faces). Opening the discovery into a bud and a stem creates a blossoming (i.e. with a bud and a stem) unicellular map. In this map, it is possible to label all the corners in their order during the tour of the unique face. Thus, there is a labelling of all the corners of the map, and a discovery vertex (we will not need the discovery itself in what follows).

The discovery vertex is obviously of degree 3 (because a leaf is adjacent to only one face). We can now introduce special vertices and *trisections*.

Definition 4.6.2. A trisection is a vertex whose corner labels are in counterclockwise order around this vertex. A vertex is said to be *special* if it is a trisection or if it is the discovery vertex.

A trisection of the map is exactly a trisection of the unicellular (blossoming) map. In [Cha10, Cha11], it is proven that in a unicellular map of genus g, there are $2g$ trisections. Furthermore, we can easily verify that the discovery vertex is not a trisection, thus the following lemma holds:

Lemma 4.6.1. *There are* $2g+1$ *special vertices in a two-faced precubic map of genus* g *(see Figure 4.13 right).*

The operation we will consider is fairly simple: take a map of genus g with a marked special vertex, and split it. There are two possible cases:

1. the map is disconnected into two maps with a marked leaf each, such that the genuses, number of edges and faces add up,

2. the resulting map is of genus $g-1$ and has 3 marked leaves, along with information on how to glue back the leaves together (given 3 leaves, there are 2 possible ways of gluing them together).

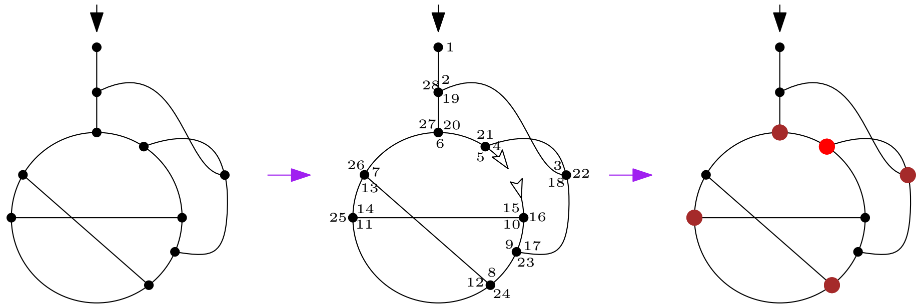


Figure 4.13 – A precubic map (left), its exploration (center), and its special points in fat (right, discovery vertex in red, trisections in brown).

Remark 4.6.2. *The first case can only appear if the special vertex is the discovery vertex. In the second case, the special vertex can be either the discovery vertex or a trisection.*

There is a bijection between maps in case 1 and pairs of maps with a marked leaf each, the inverse operation being the same as the case of disconnecting discoveries in the planar case: given two maps with a marked leaf each, it is possible to glue them back together as in Figure 4.4, and one obtains a map with a marked discovery vertex.

Case 2 is more complicated: given a map of genus $g-1$ with 3 marked leaves, that we will call a *tripod*, there are two ways to glue back the leaves. A gluing is said to be *valid* if the resulting map has genus g and the gluing vertex is a special vertex. Tripods can have 0, 1 or 2 valid gluings, and we will provide a classification of tripods with respect to their number of gluings. In the following, we will refer to "marked leaves" as just "leaves" as there is no risk of ambiguity. In order to prove (4.6.1), we will then have to prove there is a bijection between tripods with respectively 0 and 2 valid gluings.

Definition 4.6.3. In a given map, let v be the discovery vertex. It has two corners in the root face, and is thus seen twice in the tour of the root face. We can decompose the root face F as a word on side-edges as in Section 4.3, starting from the root, as $F = TvOv\overline{T}$ (see Figure 4.14 left).

Take a given tripod M, with exactly two leaves in the root face, both in $\overline{T}$. Glue those two leaves to split the root face in two. Let F' be the face thus obtained that is not the root face. We say that M is in the special case if no side-edge of F' has its opposite side-edge in T (see Figure 4.14 right).

Lemma 4.6.3. *The following classification holds:*

- *A tripod with all leaves in the same face or in the special case or with two leaves in O has 1 valid gluing.*

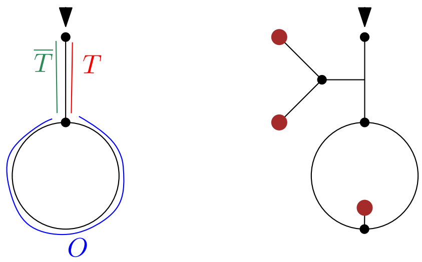


Figure 4.14 – The schematic decomposition of the root face (left), and the special case (right, with the leaves in fat brown).

- *A tripod having exactly one leaf in the root face has* 2 *valid gluings.*
- *A tripod with exactly two leaves in the root face, at least one of which in T or $\overline{T}$ (except for the special case), has no valid gluings.*

The proof of this lemma is done by carefully considering the cases implied (we omit it here).

To finish the proof, one needs to find a bijection between tripods with 0 valid gluings and tripods with 2 valid gluings. The bijection consists of cleverly "unplugging" the discovery and plugging it somewhere else to "transfer" leaves between the faces. The proof involves several cases and is a bit technical, thus we prefer to omit it. This shows that on average, a tripod has one valid gluing, implying (4.6.1).

Chapter 5

Recurrence formulas for bipartite maps with prescribed degrees (and other models)

Abstract. This chapter is adapted from the article *Simple formulas for constellations and bipartite maps with prescribed degrees*, to appear in the *Canadian Journal of Mathematics* [Lou19b]. We obtain simple quadratic recurrence formulas counting bipartite maps on surfaces with prescribed degrees (in particular, $2k$-angulations), and constellations. These formulas are the fastest known way of computing these numbers. Our work is a natural extension of previous works on integrable hierarchies (2-Toda and KP), namely the Pandharipande recursion for Hurwitz numbers (proven by Okounkov and simplified by Dubrovin–Yang–Zagier), as well as formulas for several models of maps (Goulden–Jackson, Carrell–Chapuy, Kazarian–Zograf). As for those formulas, a bijective interpretation is still to be found. We also include a formula for monotone simple Hurwitz numbers derived in the same fashion. These formulas also play a key role in establishing the hyperbolic local limit of random bipartite maps of large genus (joint work with T. Budzinski, see Chapter 7).

In Chapter 2, we recalled the proof that the generating function of constellations (and therefore, its specialization to maps) is a tau-function of the KP hierarchy. Using this fact, recurrence formulas for maps were found, starting with Goulden and Jackson for triangulations [GJ08]. They were followed by Carrell and Chapuy for general maps [CC15], and Kazarian and Zograf for bipartite maps [KZ15]. All these works start with the first KP equation, and then use ad-hoc techniques to obtain explicit recurrence formulas. The generality of this second step is not well understood. Before these works on maps, the Hurwitz numbers, that enumerate ramified coverings of the sphere, were also studied from the point of view of integrable hierarchies. Pandharipande conjectured a recurrence formula for those numbers [Pan00], which was proven by Okounkov [Oko00] and later simplified by Dubrovin, Yang and Zagier [DYZ17].

The approach developed in [GJ08, CC15, KZ15] does not generalize to constellations, nor to controlling face degrees (except for the particular minimal case of triangulations [GJ08]). On the other hand, in [Oko00, DYZ17], formulas are derived only for Hurwitz numbers unramified at 0 and ∞ (which corresponds to maps without control over the degrees of the faces and/or vertices).

We manage to combine these two approaches in the context of maps, and we derive recurrence formulas for bipartite maps with prescribed degrees, allowing us in particular to derive a formula for bipartite $2k$-angulations. We also find recurrence formulas for constellations.

These formulas are, up to our knowledge, the simplest and fastest way to compute those numbers (in all models, it takes $O(n^2g^3)$ arithmetic operations to compute the coefficient for n edges and genus g, see Remark 5.1.5). In addition to the computational aspect, such recurrence formulas are the only tool we know of in the study of asymptotic properties of large genus maps: the Goulden–Jackson formula played a key role in the proof (see Chapter 6) of the Benjamini–Curien conjecture [Cur16] of the convergence of random high genus triangulations towards a random hyperbolic map. Similarly, the results of this chapter are necessary in the study of random high genus bipartite maps with T. Budzinski (see Chapter 7).

We recall from Chapter 2 the generating function of constellations[1] (that depends implicitly on r):

$$\tau(z,\mathbf{p},\mathbf{q},(u_j)) = \sum_{\substack{n\geqslant 0\\|\mu|=|\lambda|=n\\l_i\geqslant 1\ \forall i}} \frac{z^n}{n!}\prod_{i=1}^{r} u_i^{n-l_i} p_\lambda q_\mu Cov(l_1,l_2,\ldots,l_r;\lambda,\mu)$$

(recall that the coefficients Cov are introduced in Theorem 2.1.4).

It can be written in the following form:

$$\tau(z,\mathbf{p},\mathbf{q},(u_j)) = \langle\emptyset|\Gamma_+(\mathbf{p})z^H\Lambda\Gamma_-(\mathbf{q})|\emptyset\rangle$$

with

$$F(u) = \sum_{k>0}\sum_{i=0}^{k-1/2}\log(1+ui)\psi_k\psi_k^* + \sum_{k<0}\sum_{i=0}^{-k-1/2}\log(1-ui)\psi_k^*\psi_k,$$

[1]more precisely, $r+1$-constellations where two cycles types are recorded.

where the k's belong to $\mathbb{Z}+1/2$, and $\Lambda = \prod_{j=1}^{r} \exp(F(u_j))$. Introducing the family $(\tau_n)_{n\geqslant 0}$ as

$$\tau_n(z, \mathbf{p}, \mathbf{q}, (u_j)) = \langle \emptyset_n | \Gamma_+(\mathbf{p}) z^H \Lambda \Gamma_-(\mathbf{q}) | \emptyset_n \rangle$$

we have $\tau = \tau_0$ and the τ_n's form a family of tau-functions of the 2-Toda hierarchy.

Structure of the chapter: In Section 5.1, we will give precise definitions and state our main results. The rest of the chapter presents the main steps of the proof. The first part of the proof is common to all models: in Chapter 2 we introduced τ, a certain generating function for constellations. This function, along with some auxilliary functions τ_n, satisfies the 2-Toda hierarchy. Our first contribution, inspired by [Oko00], is to link τ to the τ_n and derive an equation involving τ only (Proposition 5.2.1). This will be presented in Section 5.2. From this equation, specialized to the model we wish for (bipartite maps or constellations), we perform a few combinatorial operations (that are specific to the model, similarly as in [GJ08, CC15, KZ15]) to obtain our formulas. We present this in detail for bipartite maps in Section 5.3, and we briefly mention the case of constellations. In Section 5.4, we will present additional models, especially one-faced constellations, and in Section 5.5 we will derive a similar formula for (simple, unramified) monotone Hurwitz numbers.

5.1 Definitions and main results

We briefly recall some definitions about bipartite maps and constellations:

Definition 5.1.1. A *bipartite map* is a map with two types of vertices (black or white), such that each edge connects two vertices of different colours. A bipartite map is said to be *rooted* if a particular edge is distinguished.

An *m-constellation*[2] is a particular kind of map with two kinds of vertices: coloured vertices, carrying a "colour" between 1 and m, and star vertices. Each edge connects a star vertex to a coloured vertex. A star vertex has degree m, and its neighbours have colour 1,2,...,m in the couterclockwise cyclic order. A constellation is said to be rooted if a particular star vertex is distinguished. A constellation with n star vertices is said to be *labelled* if each star vertex carries a different label between 1 and n. Since rooting kills all possible automorphisms, there is a $(n-1)!$-to-1 correspondence between

[2] here we use the letter m instead of r, that is used in the definition of τ, because we will specialize some variables, resulting in $r = m + 1$ or $r = m + 2$.

labelled and rooted constellations with n star vertices. From now on, we will only consider rooted objects unless stated otherwise.

Some basic, well-known, properties of maps and constellations will be useful later.

Proposition 5.1.1. *Labelled (non-necessarily connected) m-constellations with n star vertices are in bijection with $(m+1)$-uples $(\sigma_1, \sigma_2, \ldots, \sigma_m, \phi)$ of permutations of $\mathfrak{S}_n$ such that $\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_m = \phi$. The permutation σ_i represents the vertices of colour i: each vertex is a cycle of σ_i, and the elements of the cycle represent the neighbouring star vertices, in that cyclic order. The permutation ϕ encodes the faces, see Figure 5.1 for an example. Bipartite maps are in bijection with 2-constellations, since each star vertex and its two adjacent edges can be merged into a single edge connecting a black and a white vertex.*

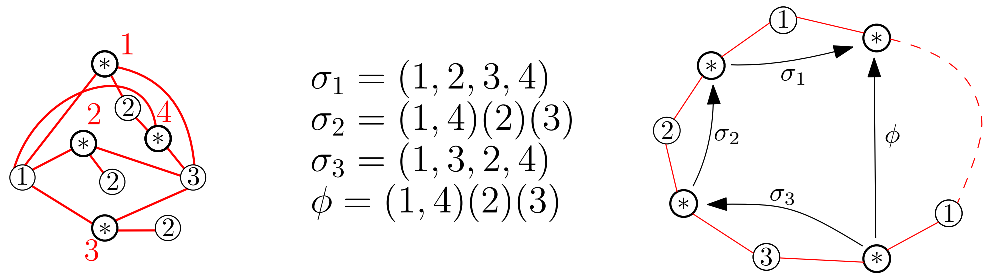


Figure 5.1 – Left: a (labelled) 3-constellation (of genus 0) and the corresponding permutations, right: the permutation ϕ, whose cycles describe the faces.

The main results of this chapter are the following theorems:

Theorem 5.1.2. *The number $\beta_g(\mathbf{f})$ of bipartite maps of genus g with f_i faces of degree $2i$ (for $\mathbf{f} = (f_1, f_2, \ldots)$) satisfies:*

$$\binom{n+1}{2}\beta_g(\mathbf{f}) = \sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f} \\ g_1+g_2+g^*=g}} (1+n_1)\binom{v_2}{2g^*+2}\beta_{g_1}(\mathbf{s})\beta_{g_2}(\mathbf{t}) + \sum_{g^* \geqslant 0}\binom{v+2g^*}{2g^*+2}\beta_{g-g^*}(\mathbf{f}) \tag{5.1.1}$$

where $n = \sum_i if_i$, $n_1 = \sum_i is_i$, $v = 2-2g+n-\sum_i f_i$, $v_2 = 2-2g_2+n_2-\sum_i t_i$ and $n_2 = \sum_i it_i$ (the n's count edges, the v's count vertices, in accordance with the Euler formula), with the convention that $\beta_g(\mathbf{0}) = 0$.

Theorem 5.1.3. *The numbers $C^{(m)}_{g,n}$ of m-constellations of genus g with n star vertices satisfy the following recurrence formula:*

$$\binom{n}{2} C^{(m)}_{g,n} = \sum_{\substack{n_1+n_2=n \\ n_1,n_2 \geqslant 1 \\ g=g_1+g_2+g^*}} n_1 \binom{(m-1)n_2+2-2g_2}{2g^*+2} C^{(m)}_{g_1,n_1} C^{(m)}_{g_2,n_2}.$$

Theorem 5.1.2 has an immediate corollary, i.e. a recurrence formula for bipartite $2k$-angulations:

Corollary 5.1.4. *The number $A^{(k)}_{g,n}$ of bipartite $2k$-angulations of genus g with n faces satisfies the following recurrence formula:*

$$\binom{kn+1}{2} A^{(k)}_{g,n} = \sum_{\substack{n_1+n_2=n \\ n_1,n_2 \geqslant 1 \\ g_1+g_2+g^*=g}} (kn_1+1) \binom{(k-1)n_2+2-2g_2}{2g^*+2} A^{(k)}_{g_1,n_1} A^{(k)}_{g_2,n_2}$$
$$+ \sum_{g^* \geqslant 0} \binom{(k-1)n+2-2(g-g^*)}{2g^*+2} A^{(k)}_{g-g^*,n}.$$

Remark 5.1.5. *Theorem 5.1.2 allows to compute the number of maps with prescribed degrees way faster than the usual Tutte–Lehman–Walsh approach [WL72, BC86, Gao93] or the topological recursion (see e.g. [Eyn16]), especially for large genus (because these methods require counting maps with up to g boundaries to enumerate maps of genus g). Note that, in order to compute the coefficients recursively, a term from the RHS has to be moved to the LHS, and we need the initial condition $\beta_0(\mathbf{1}_n) = Cat(n)$.*

We observe that Theorem 5.1.3 applies to bipartite maps (for $m = 2$). However, we have no analogue of Theorem 5.1.2 (with prescribed face degrees) for m-constellations with $m \geqslant 3$. We give a brief explanation of that fact in Remark 5.3.2.

Remark 5.1.6. *The coefficients in our recurrence formulas have a combinatorial flavor. It is a natural question to ask for a bijective proof of these formulas. However, the bijective interpretation of formulas derived from the KP/2-Toda hierarchies is still a widely open question, as bijections have only been found for certain formulas, in the particular cases of one-faced [CFF13] and planar maps (Chapter 4). Note that there is an asymmmetry in the factors in the quadratic sums, contrary to the formulas in [GJ08, CC15], but similarly to [KZ15].*

5.2 The master equation

In this section we derive the following general equation:

Proposition 5.2.1. *The general generating function of connected constellations $\mathcal{H} = \log \tau$ satisfies:*

$$\begin{aligned} D\mathcal{H}_{1,1} - \mathcal{H}_{1,1} =& \mathcal{H}_{1,1} \cdot \left(D\mathcal{H} \left(z \cdot \prod_{j=1}^{r} (1+u_j), (\frac{u_j}{1+u_j}) \right) \right) \\ &+ \mathcal{H}_{1,1} \cdot \left(D\mathcal{H} \left(z \cdot \prod_{j=1}^{r} (1-u_j), (\frac{u_j}{1-u_j}) \right) - 2D\mathcal{H} \right) \end{aligned} \tag{5.2.1}$$

with $\mathcal{H}_{1,1} = \frac{\partial^2}{\partial p_1 \partial q_1}\mathcal{H}$ and $D = z\frac{\partial}{\partial z}$.

Remark 5.2.2. *In the formula above, we omitted some of the arguments of $\mathcal{H}$. For instance, $\mathcal{H} := \mathcal{H}(z, \mathbf{p}, \mathbf{q}, (u_j))$, and*

$$\mathcal{H} \left(z \cdot \prod_{j=1}^{r} (1+u_j), (\frac{u_j}{1+u_j}) \right) := \mathcal{H} \left(z \cdot \prod_{j=1}^{r} (1+u_j), \mathbf{p}, \mathbf{q}, (\frac{u_j}{1+u_j}) \right).$$

This formula will be the starting point for all the particular cases we will consider in the next section: for each model, we will apply a particular specialization of the variables, then interpret combinatorially the operator $\frac{\partial}{\partial p_1}\frac{\partial}{\partial q_1}$ (depending on the model), and finally the extraction of coefficients will give us the relevant formulas.

We first need to relate the auxiliary functions τ_1 and τ_{-1} to the generating function τ.

Lemma 5.2.3. *The auxiliary functions τ_1 and τ_{-1} are related to τ by the following equation:*

$$\tau_{\pm 1}(z, \mathbf{p}, \mathbf{q}, (u_j)) = z^{1/2} \tau \left(z \cdot \prod_{j=1}^{r} (1 \pm u_j), \mathbf{p}, \mathbf{q}, (\frac{u_j}{1 \pm u_j}) \right).$$

Proof. We will describe how the operators behave under the action of the shift operator, then using the operator form (2.3.2) of τ we will derive the result.

It is easily verified that the operators R, C and H (as defined in Definition 2.2.3) commute with $\Gamma_+(\mathbf{p})$ and $\Gamma_-(\mathbf{q})$. We also have $R^{-n}HR^n = H + nC + \frac{n^2}{2}$.

Now we turn to the operator F (defined in Lemma 2.3.2). By a careful change of indices, we obtain

$$\begin{aligned} RFR^{-1} &= \sum_{k>0}\sum_{i=1}^{k-1/2}\log(1+ui)\psi_{k+1}\psi_{k+1}^* + \sum_{k<0}\sum_{i=1}^{-k-1/2}\log(1-ui)\psi_{k+1}^*\psi_{k+1} \\ &= \sum_{k>0}\sum_{i=0}^{k-1/2}\log(1-u+ui)\psi_k\psi_k^* + \sum_{k<0}\sum_{i=0}^{-k-1/2}\log(1-u-ui)\psi_k^*\psi_k \\ &= F(\frac{u}{1-u}) + (H-C/2)\log(1-u). \end{aligned}$$

Since $|\emptyset_{-1}\rangle = R^{-1}|\emptyset\rangle$, we have

$$\begin{aligned} \tau_{-1}(z,\mathbf{p},\mathbf{q},(u_j)) &= \langle\emptyset_{-1}|\Gamma_+(\mathbf{p})z^H\Lambda\Gamma_-(\mathbf{q})|\emptyset_{-1}\rangle \\ &= \langle\emptyset|R\Gamma_+(\mathbf{p})z^H\Lambda\Gamma_-(\mathbf{q})R^{-1}|\emptyset\rangle \\ &= \langle\emptyset|\Gamma_+(\mathbf{p})z^{RHR^{-1}}R\Lambda R^{-1}\Gamma_-(\mathbf{q})|\emptyset\rangle\,. \end{aligned}$$

But $z^{RHR^{-1}} = z^{H-C+1/2}$, and

$$R\Lambda R^{-1} = \prod_{j=1}^{r}\exp\left(F(\frac{u_j}{1-u_j}) + (H-C/2)\log(1-u_i)\right),$$

therefore

$$\begin{aligned} \tau_{-1}(z,\mathbf{p},\mathbf{q},(u_j)) &= \langle\emptyset|\Gamma_+(\mathbf{p})z^{RHR^{-1}}R\Lambda R^{-1}\Gamma_-(\mathbf{q})|\emptyset\rangle \\ &= z^{1/2}\langle\emptyset|\Gamma_+(\mathbf{p})(z\prod_{j=1}^{r}(1-u_j))^H\Lambda\left((\frac{u_j}{1-u_j})\right)\Gamma_-(\mathbf{q})|\emptyset\rangle \\ &= z^{1/2}\tau\left(z\cdot\prod_{j=1}^{r}(1-u_j),\mathbf{p},\mathbf{q},(\frac{u_j}{1-u_j})\right). \end{aligned}$$

Similarly,

$$\tau_1(z,\mathbf{p},\mathbf{q},(u_j)) = z^{1/2}\tau\left(z\cdot\prod_{j=1}^{r}(1+u_j),\mathbf{p},\mathbf{q},(\frac{u_j}{1+u_j})\right).$$

□

Remark 5.2.4. *The idea of expressing $\tau_{\pm1}$ in terms of τ by calculating $R^{\mp1}\Lambda R^{\pm1}$ is inspired by the calculation performed in [Oko00], Section 2.7.*

We can now prove Proposition 5.2.1.

Proof of Proposition 5.2.1. Recall, from Chapter 2 (more precisely (2.2.8)), that

$$\frac{\partial^2}{\partial p_1 \partial q_1}\log \tau_0 = \frac{\tau_1 \tau_{-1}}{\tau_0^2}.$$

Using Lemma 5.2.3, we obtain an equation involving τ only:

$$\frac{\partial^2}{\partial p_1 \partial q_1}\log \tau = z\frac{\tau\Big(z\cdot\prod_{j=1}^r (1+u_j), \mathbf{p}, \mathbf{q}, (\frac{u_j}{1+u_j})\Big)\tau\Big(z\cdot\prod_{j=1}^r (1-u_j), \mathbf{p}, \mathbf{q}, (\frac{u_j}{1-u_j})\Big)}{\tau^2}.$$

Substituting $\mathcal{H} = \log \tau$ in the above equation, one obtains:

$$\mathcal{H}_{1,1} = z\exp\left(\mathcal{H}(z\cdot\prod_{j=1}^r (1+u_j), (\frac{u_j}{1+u_j})) + \mathcal{H}(z\cdot\prod_{j=1}^r (1-u_j), (\frac{u_j}{1-u_j})) - 2\mathcal{H}\right). \tag{5.2.2}$$

Finally, we get (5.2.1) by applying the operator $D-1$ to both sides of (5.2.2) and getting rid of the exponential part by using (5.2.2) once more. $\square$

5.3 Proof of the main formulas

In the following subsections, we will specialize some of the variables to fit the cases we care about. To avoid tedious notations, and as there is no risk of ambiguity, the specialization of the function $\mathcal{H}$ will still be called $\mathcal{H}$.

5.3.1 Bipartite maps

In this section, we want to count bipartite maps while controlling the degrees of the faces. Hence, we will consider the case $r = 2$, and specialize $\mathcal{H}$ by setting $u_1 = u_2 = u$ and $q_i = \mathbb{1}_{i=1}$.

Let $\beta_g(\mathbf{f})$ be the number of (rooted) bipartite maps of genus g with f_i faces of degree $2i$, and $B(z, \mathbf{p}, u)$ be the ordinary generating function of connected rooted bipartite maps, defined as

$$B = \sum_{g,\mathbf{f}} z^n u^{2n-v} \prod_{i\geqslant 1} p_i^{f_i}\beta_g(\mathbf{f}).$$

with $n = \sum_i i f_i$ and $v - n + \sum_i f_i = 2 - 2g$ (Euler formula).

Equation (5.2.1) can be rewritten in terms of B only:

Lemma 5.3.1.

$$\begin{aligned}(D+1)DB =&(u^{-2}+(D+1)B)\left(B(z(1+u)^2,\mathbf{p},\frac{u}{1+u})\right)\\ &+(u^{-2}+(D+1)B)\left(B(z(1-u)^2,\mathbf{p},\frac{u}{1-u})-2B\right).\end{aligned} \tag{5.3.1}$$

Proof. In this section, $\mathcal{H}$ is the (exponential) generating function of labelled bipartite maps, and as mentioned in Definition 5.1.1, there is a $(n-1)!-to-1$ correspondence between labelled and rooted bipartite maps. Hence

$$B = D\mathcal{H}.$$

We will now express $\mathcal{H}_{1,1}$ in terms of B. The specialization $q_i = \mathbb{1}_{i=1}$ implies that only the terms $z^n q_1^n$ from the original function survived, and thus in this case

$$\frac{\partial}{\partial q_1}\mathcal{H} = D\mathcal{H}.$$

Finally, applying $\frac{\partial}{\partial p_1}$ corresponds to marking a digon. A marked digon can be contracted into a marked edge (see Figure 5.2) except when the bipartite map is just one edge, thus $\frac{\partial}{\partial p_1}\mathcal{H} = z + u^2 zD\mathcal{H} = z + u^2 zB$ (the $u^2 z$ factor comes from the fact that we lose an edge when we contract the digon, and the z term is the case where we cannot contract the digon). □

We are finally ready to prove Theorem 5.1.2.

Proof of Theorem 5.1.2. We look at the factor

$$B(z(1+u)^2,\mathbf{p},\frac{u}{1+u}) + B(z(1-u)^2,\mathbf{p},\frac{u}{1-u}) - 2B$$

in (5.3.1). The coefficient of $z^n\prod_{i\geqslant 1}p_i^{f_i}$ in it is:

$$\sum_{v>0}\beta_g(\mathbf{f})u^{2n-v}((1+u)^v+(1-u)^v-2)=\sum_{v>0}\beta_g(\mathbf{f})u^{2n-v}\left(2\sum_{0<k\leqslant\frac{v}{2}}u^{2k}\binom{v}{2k}\right).$$

In the sum above, we have, by Euler's formula, $g = \frac{n-\sum f_i - v+2}{2}$ (with the convention that $\beta_g(\mathbf{f}) = 0$ if g is not an integer). Extracting the coefficient of $z^n u^{2n-v}\prod_{i\geqslant 1}p_i^{f_i}$ in (5.3.1), one gets the result. □

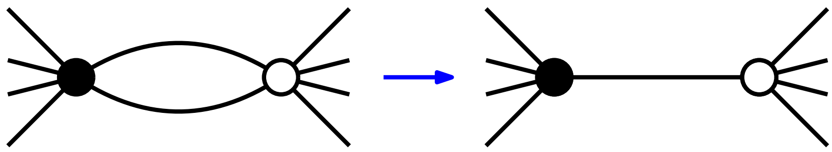


Figure 5.2 – Contracting a digon.

5.3.2 Constellations

In this section, we will count constellations without controlling the degrees of the faces. For that, we will specialize $\mathcal{H}$ by taking $r = m+1$, $p_i = q_i = \mathbb{1}_{i=1}$, and $u_i = u$ for all i. The variable u counts the number of coloured vertices plus the number of faces, or equivalently, by Euler's formula, the genus.

Proof of Theorem 5.1.3. After the specialization, $\mathcal{H}_{1,1}$ becomes $D^2\mathcal{H}$ by the same argument as in the proof of Theorem 5.1.2. If we take C to be the (ordinary) generating function of connected constellations, i.e.

$$C = \sum_{g,n} z^n u^{2n+2g-2} C^{(m)}_{g,n},$$

we have, as before, $C = D\mathcal{H}$. Equation (5.2.1) becomes

$$(D^2 - D)C = DC\Big(C\Big(z(1+u)^{m+1}, \frac{u}{1+u}\Big) + C\Big(z(1-u)^{m+1}, \frac{u}{1-u})\Big) - 2C\Big). \tag{5.3.2}$$

To finish the proof, we proceed exactly as in the proof of Theorem 5.1.2: first compute the coefficient of z^n in

$$C\left(z(1+u)^{m+1}, \frac{u}{1+u}\right) + C\left(z(1-u)^{m+1}, \frac{u}{1-u})\right) - 2C,$$

then just extract the coefficient of $z^n u^{2n+2g-2}$ in (5.3.2) (the suitable exponent of u is derived by the Euler formula). ☐

Remark 5.3.2. *This time, we cannot track the degrees of the faces, as in general the combinatorial operation of contracting an m-gon might disconnect the map, and the formula gets messy. However, if we restrict to only one face we can perform this operation to recover a nice formula (see Section 5.4).*

5.4 Additional results

5.4.1 One-faced constellations

In this section, we will derive a recurrence formula for constellations with one face. In the case of bipartite maps, the formula is just a particular case of (5.1.1), but for $m \geqslant 3$, it cannot be derived from Theorem 5.1.3 directly. One-faced constellations were first enumerated in [PS02]: an exact formula given the degree distribution of each coloured vertex is provided. While the following formula does not give control over the degrees of the vertices, it is much quicker to compute the "global" (i.e. controlling only the genus and the number of vertices) number of one-faced constellations for $m \geqslant 3$ (for $m = 2$, i.e. bipartite maps, a nice formula for one-faced bipartite maps can be found in [Adr97]).

Theorem 5.4.1. *Let $U_m(g,n)$ be the number of one-faced m-constellations of genus g with n star vertices. Also, let $U_m^{(k)}(g,n)$ be the number of one-faced m-constellations of genus g with n star vertices and k distinguished (pairwise distinct) coloured vertices, i.e. $U_m^{(k)}(g,n) = \binom{(m-1)n+1-2g}{k} U_m(g,n)$. We have the following recurrence formula:*

$$\frac{n(n+1)^{m-1}}{2} U_m(g,n) = \sum_{g^*=0}^{g} U_m^{(2+2g^*)}(g-g^*,n). \tag{5.4.1}$$

Remark 5.4.2. *This formula reminds of the formula for one-faced maps proven bijectively by Chapuy in [Cha11]. Indeed, it allows to compute the number of one-faced maps of genus g in terms of number of maps of lower genus with the same number of edges and some distinguished vertices. The difference, though, is that in Chapuy's formula there are an odd number of distinguished vertices, whereas in (5.4.1) there are an even number of distinguished vertices.*

Nevertheless, there might be a connection as those formulas arise in the same algebraic context. Our formula is obtained via the 2-Toda hierarchy, whereas Chapuy's is an intermediate step to prove the Harer–Zagier recurrence formula (see [CFF13]), which is itself a special case of a formula obtained via the KP hierarchy: the Carrell–Chapuy recurrence formula [CC15].

To prove (5.4.1), we will take $r = m$ and apply the following specialization to (5.2.1): fix an integer n, and set $q_i = \mathbb{1}_{i=1}$, $u_i = u$ for all i, as well as $z = 1$. Set also $p_i = 0$ for all $i \neq n$, and extract the coefficient of p_n^1. Now $\mathcal{H}$ is simply a polynomial in u. It counts labelled one-faced constellations. Let U be the associated polynomial for rooted objects, the classical correspondence between labelled and rooted objects yields $U = D\mathcal{H}$. As before, there is a "marked m-gon", and we need to interpret this combinatorially:

Lemma 5.4.3. *After applying the specialization, the LHS of (5.2.1) becomes $n(n+1)^{m-1}U$.*

Proof. The only terms of $\mathcal{H}_{1,1}$ in (5.2.1) that survive the specialization are the coefficients of $z^{n+1}p_n p_1 q_1^{n+1}$. Therefore we have $D\mathcal{H}_{1,1} = (n+1)\mathcal{H}_{1,1}$ and $\mathcal{H}_{1,1} = (n+1)\frac{\partial}{\partial p_1}\mathcal{H}$. The LHS of (5.2.1) is therefore equal (after specialization) to $n(n+1)\frac{\partial}{\partial p_1}\mathcal{H}$. It remains to show that $\frac{\partial}{\partial p_1}\mathcal{H} = (n+1)^{m-2} \cdot U$.

Applying $\frac{\partial}{\partial p_1}$ corresponds to marking an m-gon. As in the proof of Theorem 5.1.2, it kills all symmetries, thus there is a $(n+1)!$-to-1 correspondence between labelled constellations and constellations with a marked m-gon. Therefore, $\frac{\partial}{\partial p_1}\mathcal{H}$ is the ordinary generating function of connected unlabelled m-constellations with one face of degree mn and one face of degree m.

We will work with permutations to make things easier. Connected unlabelled m-constellations with one face of degree mn and one face of degree m are in bijection with $(m+1)$-uples of permutations $\phi, \sigma_1, \dots \sigma_m$ of $\mathfrak{S}_{n+1}$ satisfying the following constraints:

- $\phi = \prod_{i=1}^{m} \sigma_i$,
- In cycle products, ϕ is written $(1, 2, \dots, n)(n+1)$,
- The image of 1 by σ_1 is $n+1$.

We can describe the operation of "contracting an m-gon" on the permutations. To $\phi, \sigma_1, \dots \sigma_m$ we will associate a $(m+1)$-uple $\phi', \sigma'_1, \dots \sigma'_m$ of permutations of $\mathfrak{S}_n$:

- To ϕ, we associate $\phi' = (1, 2, \dots, n)$,
- For $1 \leqslant i < m$, to σ_i we associate the permutation σ'_i where in the cycle product we just deleted the element $n+1$ (see Figure 5.3).
- To σ_m we associate $\sigma'_m = \phi' \left(\prod_{i=1}^{m-1} \sigma'_i\right)^{-1}$.

This exactly describes a rooted m-constellation with one face of degree mn. To go back, one needs to remember, for $1 < i < m$, what was the preimage of $n+1$ in σ_i (including possibly $n+1$ itself). There are $n+1$ possible choices for each i, thus after the specialization, $\frac{\partial}{\partial p_1}\mathcal{H} = (n+1)^{m-2} \cdot U$. □

A simple calculation in the right-hand side finishes the proof:

Proof of Theorem 5.4.1. In the RHS, we have a product of two terms. Since $\mathcal{H}$ has no constant coefficient in the p_i's, after specialization we get the coefficient of p_n^0 of $\mathcal{H}_{1,1}$ (which is just u^{m-1}, corresponding to the constellation

$$(1,2,5,7)(3,8)(4,6) \rightarrow (1,2,5,7)(3)(4,6)$$

$$(1,5)(3,7,4,2,6)(8) \rightarrow (1,5)(3,7,4,2,6)$$

Figure 5.3 – Deleting $n+1$, for $n=7$, whether $n+1$ is a fixed point or not.

with only one star vertex) times the coefficient of p_n^1 in

$$D\mathcal{H}\left((1+u)^m, \frac{u}{1+u}\right) + D\mathcal{H}\left((1-u)^m, \frac{u}{1-u}\right) - 2D\mathcal{H}.$$

Again, since $D\mathcal{H} = U$, we can extract the coefficient of $z^n u^{mn-v}$ (where $v = (m-1)n+1-2g$ by Euler's formula), as in the proof of Theorem 5.1.2, and obtain the result. □

5.4.2 Controlling more parameters

In each of the previous cases, we specialized a lot of variables to obtain formulas for "global" coefficients. Starting over from (5.2.1) without specializing some of the variables, one is able to obtain (slightly more complicated) formulas for more fine-grained coefficients. As an example, we can compute the number $C^{(m)}_{g,n,f}$ of m-constellations of genus g, with n star vertices and f faces:

$$\binom{n}{2} C^{(m)}_{g,n,f} = \sum n_1 \binom{f_2}{k} \binom{2g_2 - f_2 + (m-1)n_2}{2g^* + 2 - k} C^{(m)}_{g_1,n_1,f_1} C^{(m)}_{g_2,n_2,f_2} \tag{5.4.2}$$

where the sum is over $n_1 + n_2 = n$, $n_1, n_2 > 0$, $g^* \geqslant 0$, $g_1 + g_2 + g^* = g$ and $f_1 + f_2 - k = f$.

The proof of Theorem (5.4.2) is essentially the same as the proof of Theorem 5.1.3, except that we do not specialize $u_i = u$ for all i, but only for $i \leqslant m$. In this case, u counts coloured vertices, and u_{m+1} counts faces.

Remark 5.4.4. *Even though the summation is complicated,* (5.4.2) *allows to compute all the coefficients $C^{(m)}_{g,n,f}$ from the initial condition $C^{(m)}_{g,1,f} = 1$ iff $g = 0$ and $f = 1$, and 0 otherwise.*

However, it does not restrict to a formula for one-faced constellations.

We can also find formulas for other models, with other specializations. Relevant models include bipartite maps (with prescribed face degrees), one-faced constellations, or (general) constellations, with control over the number of vertices of each colour. We can also obtain a formula for triangulations (by specializing $r = 1$, $p_i = \mathbb{1}_{i=2}$, $q_i = \mathbb{1}_{i=3}$), but it is more complicated (and less "combinatorial") than the Goulden–Jackson formula [GJ08]. The reader is encouraged to play with (5.2.1) to find other nice formulas.

5.4.3 Univariate generating series

A relevant corollary of our results is that the formulas we obtain allow to compute the univariate generating series of some given models of maps ($2k$-angulations counted by faces, constellations counted by star vertices, etc.). To illustrate this fact, fix an integer k and let $F_g(z)$ be the generating series of genus g bipartite $2k$-angulations:

$$F_g(z) = \sum_{n>0} A_{g,n}^{(k)} z^n$$

with the coefficients $A_{g,n}^{(k)}$ as defined in Corollary 5.1.4. Our formula gives an algorithm to compute every F_g for $g \geqslant 1$, given F_0. Indeed, take $g \geqslant 1$, Corollary 5.1.4 rewrites

$$\Delta F_g = \phi(z, F_0, F_1, \ldots, F_{g-1}) \tag{5.4.3}$$

with

$$\Delta = \binom{kD+1}{2} - \left(\binom{(k-1)D+2}{2} F_0\right)(kD+1) \\ - ((kD+1)F_0)\binom{(k-1)D+2}{2} - \binom{(k-1)D+2}{2},$$

where $D = z\frac{\partial}{\partial z}$, and ϕ is a polynomial in its variables and their (first and second) derivatives. It is well known (see for instance [BDFG04]) that

$$F_0 = t - z\binom{2k-1}{k+1} t^{k+1} - 1$$

with the change of variable

$$t = 1 + z\binom{2k-1}{k} t^k.$$

Note that we have a "-1" in the expression of F_0 because we do not count the "empty map".

Assuming we know F_h for $h < g$, this gives a linear, second order ODE in F_g (with respect to the variable t). Since all the F_g's are rational in t (see for instance [CF16]), all the coefficients of the equation are themselves rational, and the solutions can be computed explicitly. The initial conditions are given by the two following facts: $[z^0]F_g = 0$ and $[z^1]F_g$ is the number of unicellular bipartite maps of genus g with k edges, that can for instance be computed using Theorem 5.4.1.

5.5 Monotone Hurwitz numbers

In this section, we derive a recurrence formula for monotone Hurwitz numbers, in a similar fashion as in previous sections. These numbers, which appear in the calculation of the HCIZ integral, were introduced in [GGPN14].

Definition 5.5.1. For two transpositions of $\mathfrak{S}_n$, we say that $(i, j) \preceq (k, l)$ if $\max(i, j) \leqslant \max(k, l)$. The double monotone Hurwitz number $\vec{H}^{\lambda,\mu}_{g,n}$ is $\frac{1}{n!}$ times the number of tuples $(t_1, t_2, \ldots, t_r, \sigma_\lambda, \sigma_\mu)$ of permutations of S_n such that:

- $r = l(\lambda) + l(\mu) + 2g - 2$ where $l(\lambda)$ is the number of parts of λ,
- $t_1, t_2, \ldots, t_r$ is an increasing sequence of transpositions,
- σ_λ (resp. σ_μ) has cycle type λ (resp. μ),
- $t_1 \cdot t_2 \cdot \ldots \cdot t_r = \sigma_\lambda \sigma_\mu$,
- the permutations $t_1, t_2, \ldots, t_r, \sigma_\lambda$ act transitively on $1, 2, \ldots, n$.

The simple monotone Hurwitz numbers $\vec{H}^{\lambda}_{g,n}$ are defined as $\vec{H}^{\lambda}_{g,n} = \vec{H}^{\lambda,1^n}_{g,n}$.

We will set $W^{\lambda,\mu}_{g,n}$ to be the same numbers without the transitivity condition, and introduce

$$\tau(z, \mathbf{p}, \mathbf{q}, u) = \sum_{\substack{n \geqslant 0 \\ |\lambda|=|\mu|=n \\ r \geqslant 0}} \frac{z^n}{n!} p_\lambda q_\mu u^r W^{\lambda,\mu}_{g,n}$$

with g such that $r = l(\lambda) + l(\mu) + 2g - 2$. Let $\mathcal{H} = \log \tau$ be the generating function of the $\vec{H}^{\lambda,\mu}_{g,n}$.

As before, it can be shown (see for instance [GPH15]) that

$$\tau(z, \mathbf{p}, \mathbf{q}, u) = \langle \emptyset | \Gamma_+(\mathbf{p}) z^H \Lambda \Gamma_-(\mathbf{q}) | \emptyset \rangle$$

with $\Lambda = \exp(-F(-u))$, where F is the function defined in Lemma 2.3.2. A general equation similar to (5.2.1) can be derived:

$$D\mathcal{H}_{1,1} - \mathcal{H}_{1,1} = \mathcal{H}_{1,1} \left(D\mathcal{H}\left(\frac{z}{1+u}, \frac{u}{1+u}\right) + D\mathcal{H}\left(\frac{z}{1-u}, \frac{u}{1-u}\right) - 2D\mathcal{H} \right) \tag{5.5.1}$$

with $\mathcal{H}_{1,1} = \frac{\partial^2}{\partial p_1 \partial q_1} \mathcal{H}$ and $D = z\frac{\partial}{\partial z}$.

Similarly as with constellations, in general we cannot even track the cycle type of σ_λ, although, from the specialization $p_i = q_i = \mathbb{1}_{i=1}$ for all i, we can obtain a recurrence formula for the unramified monotone Hurwitz numbers $\vec{H}_{g,n} = \vec{H}^{1^n}_{g,n}$:

$$n\binom{n}{2}\vec{H}_{g,n} = \sum_{\substack{n_1+n_2=n \\ g^* \geqslant 0 \\ g_1+g_2+g^*=g}} n_1^2 n_2 \binom{3n_2+2g_2+2g^*-1}{2g^*+2} \vec{H}_{g_1,n_1}\vec{H}_{g_2,n_2}. \tag{5.5.2}$$

Remark 5.5.1. *Here, the number $\vec{H}^{\lambda,\mu}_{g,n}$ are defined with a scaling factor of $\frac{1}{n!}$ to make the formula simpler; this is a different convention as in [GGPN14]. Formula (5.5.2) allows to compute all the $\vec{H}_{g,n}$ only knowing that $\vec{H}_{0,1} = 1$.*

Chapter 6

Local limits of high genus triangulations

Abstract. This chapter is adapted from the article *Local limits of high genus triangulations*, with Thomas Budzinski (submitted) [BL19]. We prove a conjecture of Benjamini and Curien stating that the local limits of uniform random triangulations whose genus is proportional to the number of faces are the Planar Stochastic Hyperbolic Triangulations (PSHT) defined in [Cur16]. The proof relies on a combinatorial argument and the Goulden–Jackson recurrence relation to obtain tightness, and probabilistic arguments showing the uniqueness of the limit. As a consequence, we obtain asymptotics up to subexponential factors on the number of triangulations when both the size and the genus go to infinity.

As a part of our proof, we also obtain the following result of independent interest: if a random triangulation of the plane T is weakly Markovian in the sense that the probability to observe a finite triangulation t around the root only depends on the perimeter and volume of t, then T is a mixture of PSHT.

The goal of this chapter is to prove the conjecture of Benjamini and Curien concerning the local limit of high genus triangulations. The only previous results on high genus maps are the identification of the local limit of uniform unicellular maps (i.e. maps with one face) [ACCR13], which is a supercritical random tree, and the calculation of their diameter [Ray15].

Let us first explain how the local limit is affected by the genus. By the Euler formula, a triangulation with $2n$ faces and genus g has $3n$ edges and $n+2-2g$ vertices. Hence, if $\frac{g}{n} \to \theta \in \left[0, \frac{1}{2}\right]$, then the average degree of the vertices goes to $\frac{6}{1-2\theta}$. In particular, if $0 < \theta < \frac{1}{2}$, this mean degree lies strictly between 6 and $+\infty$. Therefore, it is natural to expect limit objects to be hyperbolic triangulations of the plane[1].

We recall the basic properties of the PSHT constructed by Curien[2] in [Cur16]: there is a one-parameter family $(\mathbb{T}_\lambda)_{0<\lambda \leqslant \lambda_c}$ of random triangulations of the plane, where $\lambda_c = \frac{1}{12\sqrt{3}}$, that are the only random triangulations of the plane exhibiting a natural spatial Markov property. For any finite triangulation t with a hole of perimeter p and v vertices in total, we have

$$\mathbb{P}\left(t \subset \mathbb{T}_\lambda\right) = C_p(\lambda)\lambda^v,$$

where $C_p(\lambda)$ are explicit functions of λ. Moreover, $\mathbb{T}_{\lambda_c}$ is the UIPT, whereas for $\lambda < \lambda_c$, the map $\mathbb{T}_\lambda$ has hyperbolicity properties such as exponential volume growth [Ray14, Cur16], positive speed of the simple random walk [Cur16, ANR16] or the existence of a lot of infinite geodesics quickly escaping away from each other [Bud18c].

For any $g \geqslant 0$ and $n \geqslant 2g-1$, we denote by $\mathcal{T}(n,g)$ the set of rooted triangulations of genus g with $2n$ faces. By *rooted*, we mean that the triangulation is equipped with a distinguished oriented edge called the *root*. Let also $T_{n,g}$ be a uniform triangulation of $\mathcal{T}(n,g)$. We also recall that a sequence of rooted triangulations (t_n) converges locally to a triangulation T if for any $r \geqslant 0$, the ball of radius r around the root in t_n, seen as a map, converges to the ball of radius r in T. We refer to Section 6.1 for more precise definitions. For any $\lambda \in (0, \lambda_c]$, let $h \in \left(0, \frac{1}{4}\right]$ be such that $\lambda = \frac{h}{(1+8h)^{3/2}}$, and let

$$d(\lambda) = \frac{h \log \frac{1+\sqrt{1-4h}}{1-\sqrt{1-4h}}}{(1+8h)\sqrt{1-4h}}. \tag{6.0.1}$$

It can be checked that the function $d(\lambda)$ is increasing with $d(\lambda_c) = \frac{1}{6}$ and $\lim_{\lambda \to 0} d(\lambda) = 0$ (see the end of Section 6.4.3 for a quick proof). Then our main result is the following.

Theorem 6.0.1. *Let (g_n) be a sequence such that $\frac{g_n}{n} \to \theta$ with $\theta \in \left[0, \frac{1}{2}\right)$. Then we have*

$$T_{n,g_n} \xrightarrow[n\to+\infty]{(d)} \mathbb{T}_\lambda$$

for the local topology, where λ is the unique solution of the equation

$$d(\lambda) = \frac{1-2\theta}{6}. \tag{6.0.2}$$

[1] As a deterministic example, the d-regular triangulations of the plane for $d > 6$ are hyperbolic.

[2] To be exact, the triangulations defined in [Cur16] are type-II triangulations, i.e. triangulations with no loop joining a vertex to itself. The type-I (with loops) analog, which will be the one considered in this work, was defined in [Bud18b].

This result was conjectured by Benjamini and Curien [Cur16] without an explicit formula for $d(\lambda)$, and the formula for $d(\lambda)$ was first conjectured in [Bud18a, Appendix B]. The reason why this formula appears is that $d(\lambda)$ is the expected inverse of the root degree in $\mathbb{T}_\lambda$, while the corresponding quantity in T_{n,g_n} is asymptotically $\frac{1-2\theta}{6}$ by the Euler formula. While it may seem counter-intuitive that high genus objects yield planar maps in the local limit, this has already been proved for other models such as random regular graphs or unicellular maps [ACCR13]. Note that the case $\theta = 0$ corresponds to $\lambda = \lambda_c$, which proves that if $g_n = o(n)$, then T_{n,g_n} converges to the UIPT, which also seems to be a new result. On the other hand, when $\theta \to \frac{1}{2}$, we have $\lambda \to 0$, so all the range $(0, \lambda_c]$ is covered. Since the object $\mathbb{T}_0$ is not well-defined (it corresponds to a "triangulation" where the vertex degrees are infinite), we expect that if $\theta = \frac{1}{2}$, the sequence (T_{n,g_n}) is not tight for the local topology.

Strategy of the proof. The most natural idea to prove Theorem 6.0.1 would be to obtain precise asymptotics for the numbers $\tau(n,g) = |\mathcal{T}(n,g)|$ and to adapt the ideas of [AS03]. In theory, these numbers are entirely characterized by the Goulden–Jackson recurrence equation [GJ08]. However, this seems very difficult without any a priori estimate on the $\tau(n,g)$ and all our efforts to extract asymptotics when $\frac{g}{n} \to \theta > 0$ from these relations have failed. Therefore, our proof relies on more probabilistic considerations. It is however interesting to note that our probabilistic arguments allow in the end to obtain asymptotic enumeration (Theorem 6.0.3).

The first part of the proof consists of a tightness result: we prove that (T_{n,g_n}) is tight for the local topology as long as $\frac{g_n}{n}$ stays bounded away from $\frac{1}{2}$. A key tool in the proof is the *bounded ratio lemma* (Lemma 6.2.1), which states that the ratio $\frac{\tau(n+1,g)}{\tau(n,g)}$ is bounded as long as $\frac{g_n}{n}$ stays bounded away from $\frac{1}{2}$. This is essentially enough to adapt the argument of Angel and Schramm [AS03] for the tightness of $T_{n,0}$. Along the way, we also show that any subsequential limit is a.s. planar and one-ended. The Goulden–Jackson formula also plays an important role in the proof.

The next step is to notice that any subsequential limit T satisfies a weak Markov property: if t is a finite triangulation with a hole of perimeter p and v vertices in total, then $\mathbb{P}(t \subset T)$ only depends on p and v. From here, we deduce that T must be a mixture of PSHT, i.e. a PSHT with a random parameter Λ.

Finally, what is left to prove is that Λ is deterministic, i.e. it does not depend on T_{n,g_n}. By a surgery argument on finite triangulations that we call the *two holes argument*, we first show that if T_{n,g_n} is fixed, then Λ does not depend on the choice of the root. We conclude by using the fact that the average inverse degree of the root in T_{n,g_n} is asymptotically $\frac{1-2\theta}{6}$.

Weakly Markovian triangulations. Since one of the steps of the proof is a result of independent interest, let us highlight it right now. We call a random triangulation of the plane T *weakly Markovian* if for any finite triangulation t with a hole of perimeter p and v vertices in total, the probability $\mathbb{P}(t \subset T)$ only depends on p and v. This is strictly weaker than the spatial Markov property considered in [Cur16] to define the PSHT, since any mixture of PSHT is weakly Markovian. The result we prove is the following.

Theorem 6.0.2. *Any weakly Markovian triangulation of the plane is a mixture of PSHT.*

Asymptotic enumeration. Finally, while we were unable to directly obtain asymptotics on $\tau(n,g)$ when both n and g go to $+\infty$, Theorem 6.0.1 allows us to obtain such estimates up to sub-exponential factors. For any $\theta \in \left[0, \frac{1}{2}\right)$, we denote by $\lambda(\theta)$ the value of λ given by (6.0.2).

Theorem 6.0.3. *Let (g_n) be a sequence such that $0 \leqslant g_n \leqslant \frac{n+1}{2}$ for every n and $\frac{g_n}{n} \to \theta \in \left[0, \frac{1}{2}\right]$. Then we have*

$$\tau(n, g_n) = n^{2g_n} \exp\left(f(\theta)n + o(n)\right)$$

as $n \to +\infty$, where $f(0) = \log 12\sqrt{3}$, also $f(1/2) = \log \frac{6}{e}$ and

$$f(\theta) = 2\theta \log \frac{12\theta}{e} + \theta \int_2^{1/\theta} \log \frac{1}{\lambda(1/t)} \mathrm{d}t \tag{6.0.3}$$

for $0 < \theta < \frac{1}{2}$.

To the best of our knowledge, these are the first asymptotic results on the number of triangulations with both large size and high genus. Note that the integral is well-defined since $\lambda(\theta)$ is a continuous function and we have $\lambda(\theta) = O\left(1/2 - \theta\right)$ when $\theta \to 1/2$. Moreover, since $\lambda(\theta) \to \frac{1}{12\sqrt{3}}$ as $\theta \to 0$, it is easy to see that the function f is continuous at 0 and at 1/2. The proof mostly relies on the observation that Theorem 6.0.1 gives the limit values of the ratio $\frac{\tau(n+1,g)}{\tau(n,g)}$.

Structure of the chapter. In Section 6.1, we review basic definitions and previous results that will be used throughout the chapter. In Section 6.2, we prove that the triangulations T_{n,g_n} are tight for the local topology, and that any subsequential limit is a.s. planar and one-ended. In Section 6.3, we prove Theorem 6.0.2, which implies that any subsequential limit of T_{n,g_n} is a PSHT with random parameter Λ. In Section 6.4, we conclude the proof of Theorem 6.0.1 by showing that Λ is deterministic and depends only on θ. Finally, Section 6.5 is devoted to the proof of Theorem 6.0.3.

6.1 Preliminaries

6.1.1 Definitions

The goal of this paragraph is to state basic definitions and introduce notations on triangulations that will be used in all the chapter.

Recall that a *triangulation* is a rooted map where all the faces have degree 3. We will mostly be interested in *type-I triangulations*, i.e. triangulations that may contain loops and multiple edges. We mention right now that a *type-II triangulation* is a triangulation that may contain multiple edges, but no loops.

For every $n \geqslant 1$ and $g \geqslant 0$, we will denote by $\mathcal{T}(n,g)$ the set of triangulations of genus g with $2n$ faces (the number of faces must be even to glue the edges two by two). By the Euler formula, a triangulation of $\mathcal{T}(n,g)$ has $3n$ edges and $n+2-2g$ vertices. In particular, the set $\mathcal{T}(n,g)$ is nonempty if and only if $n \geqslant 2g-1$. We will also denote by $\tau(n,g)$ the cardinality of $\mathcal{T}(n,g)$ and by $T_{n,g}$ a uniform random variable on $\mathcal{T}(n,g)$.

We will also need to consider two different notions of triangulations with boundaries, that we call *triangulations with holes* and *triangulations of multi-polygons*. Basically, the first ones will be used to describe a neighbourhood of the root in a triangulation, and the second ones to describe the complementary of this neighbourhood.

For $\ell \geqslant 1$ and $p_1, p_2, \dots, p_\ell \geqslant 1$, we call a *triangulation with holes of perimeter* $p_1, \dots, p_\ell$ a map where all the faces have degree 3 except, for every $1 \leqslant i \leqslant \ell$, a face h_i of degree p_i. The faces h_i are called the *holes*. The boundaries of the faces h_i must be simple and edge-disjoint, but may have common vertices (see the bottom part of Figure 6.1). A triangulation with holes will be rooted at a distinguished oriented edge, which may lie on the boundary of a hole or not. Triangulations with holes will always be finite.

A (possibly infinite) *triangulation of the* $(p_1, \dots, p_\ell)$*-gon* is a map where all the faces have degree 3 except, for every $1 \leqslant i \leqslant \ell$, a face f_i of degree

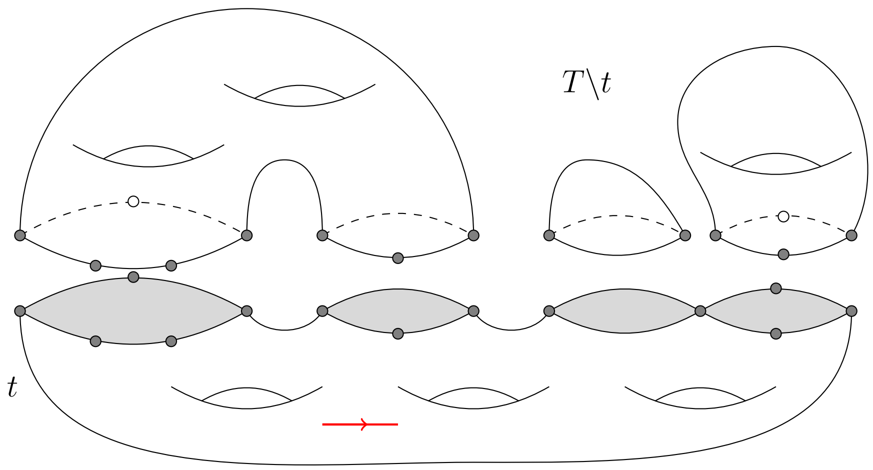


Figure 6.1 – A large triangulation T with genus 7 and a smaller triangulation with holes t (in the bottom) such that $t \subset T$. The holes are filled by a triangulation of the $(3, 5)$-gon, a triangulation of the 2-gon and a triangulation of the 4-gon. The triangles are not drawn on the picture.

p_i. The faces f_i are called the *external faces*, and must be simple and have vertex-disjoint boundaries. Moreover, each of the external faces comes with a distinguished edge on its boundary, such that the external face lies on the right of the distinguished edge.

Let $\mathcal{T}_{p_1,p_2,\dots,p_\ell}(n, g)$ be the set of triangulations of the $(p_1, p_2, \dots, p_\ell)$-gon of genus g with $2n-\sum_{i=1}^{\ell}(p_i-2)$ triangles, and by $\tau_{p_1,p_2,\dots,p_\ell}(n, g)$ its cardinality. The reason why we choose this convention is that by the Euler formula, a triangulation of $\mathcal{T}_{p_1,p_2,\dots,p_\ell}(n, g)$ has $n + 2 - 2g$ vertices in total, just like a triangulation of $\mathcal{T}(n, g)$.

If t is a triangulation with holes and T a (finite or infinite) triangulation, we write $t \subset T$ if T can be obtained from t by gluing one or several triangulations of multi-polygons to the holes of t (see Figure 6.1). In particular, in the planar case, this definition coincides with the one used e.g. in [AS03]. If T is an infinite triangulation, we say that it is *one-ended* if for every finite t with $t \subset T$, only one connected component of $T\backslash t$ contains infinitely many triangles. We also say that T is *planar* if every finite t with $t \subset T$ is planar.

We also recall that to a triangulation t, we can naturally associate its *dual map* t^*. If t is a triangulation of a multi-polygon, it will be more suitable to work with the convention that the external faces *do not* belong to the dual t^*. Note that triangulations of multi-polygons have simple and disjoint boundaries, so their dual t^* will always be connected.

Finally, we recall the definition of the *graph distance* in a map t. For a pair of vertices (v, v'), the distance $d_t(v, v')$ is the length of the shortest path of edges of t between v and v'. We call d_t^* the graph distance in the dual[3] map t^*. We also note that there is a natural way to extend d_t^* to the vertices of t. For a pair of distinct vertices (v, v'), we set

$$d_t^*(v, v') = \min(d_t^*(f, f')) + 1,$$

where the minimum is taken over all pairs (f, f') of faces such that f is adjacent to v and f' is adjacent to v', and $d_t^*(v, v) = 0$ for all v.

6.1.2 Combinatorics

The goal of this paragraph is to summarize some previously known or easy combinatorial results about triangulations in higher genus. We start with the Goulden–Jackson recurrence formula.

Theorem 6.1.1. *[GJ08] Let $f(n, g) = (3n + 2)\tau(n, g)$, with the convention $f(-1, 0) = \frac{1}{2}$ and $f(-1, g) = 0$ for $g \geqslant 1$. For every $n, g \geqslant 0$ with $g \leqslant \frac{n+1}{2}$, we have*

$$f(n, g) = \frac{4(3n + 2)}{n + 1} \left(n(3n - 2)f(n - 2, g - 1) + \sum_{\substack{n_1+n_2=n-2 \\ g_1+g_2=g}} f(n_1, g_1)f(n_2, g_2) \right). \tag{6.1.1}$$

We also state right now a crude inequality that bounds the number of triangulations of multi-polygons with genus g by the number of triangulations of genus g. This will be useful later.

Lemma 6.1.2. *For every $n, g \geqslant 0$ and $p_1, \dots, p_\ell \geqslant 1$, we have*

$$\tau_{p_1,\dots,p_\ell}(n, g) \leqslant (6n)^{\ell-1}\tau(n, g).$$

Proof. We describe a way to associate to each map t of $\mathcal{T}_{p_1,\dots,p_\ell}(n, g)$ a map $\widetilde{t}$ of $\mathcal{T}(n, g)$ with some marked oriented edges. For each external face f_i of t:

- if $p_i \geqslant 3$, we triangulate f_i by joining all the vertices of ∂f_i to the start of the distinguished edge on ∂f_i, and we mark this edge as e_i;

[3]In particular, if t is a triangulation of a multi-polygon, then $d_t^*(f, f')$ is the length of the smallest dual path which *avoids* the external faces.

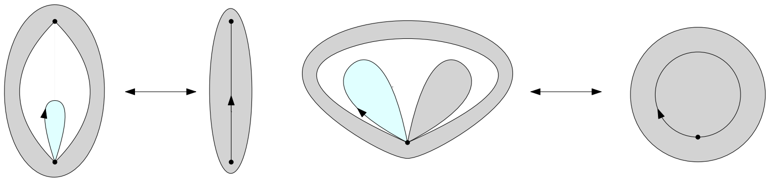


Figure 6.2 – Getting rid of a boundary of size 1. The boundary faces are in blue

- if $p_i = 2$, we simply glue together the two edges of ∂f_i, and mark the edge that we obtain as e_i;
- if $p_i = 1$, we use the "classical" root transformation shown on Figure 6.2, and mark the edge obtained by the gluing as e_i.

We obtain a triangulation with the same genus as the initial one, and we root it at e_1. Note that the above operation does not change the number of vertices, so the triangulation belongs to $\mathcal{T}(n,g)$. It is easy to see that $t \to \widetilde{t}$ is injective. Indeed, if we know $p_i \geqslant 3$ and the edge e_i, then the $p_i - 2$ triangles created by triangulating f_i are the first $p_i - 2$ triangles on the right of e_i that are incident to its starting point. If $p_i \in \{1,2\}$, the reverse operation is straightforward. Finally, $\widetilde{t}$ is a triangulation of $\mathcal{T}(n,g)$ with $\ell - 1$ marked oriented edges (plus its root edge). Since any triangulation of $\mathcal{T}(n,g)$ has $6n$ oriented edges, we are done.

□

Remark 6.1.3. *The bijection of Figure 6.2 is classical and implies in particular $\tau_1(n,g) = \tau(n,g)$.*

6.1.3 The PSHT

In this subsection, we recall the definition and some basic properties of the type-I Planar Stochastic Hyperbolic Triangulations, or PSHT. They were introduced in [Cur16] in the type-II setting (no loops), but we will be more interested in the type-I PSHT defined in [Bud18b]. The PSHT $(\mathbb{T}_\lambda)_{0<\lambda\leqslant\lambda_c}$ are a one-parameter family of random infinite triangulations of the plane, where $\lambda_c = \frac{1}{12\sqrt{3}}$. Their distribution is characterized as follows. There is a family of constants $(C_p(\lambda))_{p\geqslant 1}$ such that for every planar triangulation with a hole of perimeter p and v vertices in total, we have

$$\mathbb{P}\left(t \subset \mathbb{T}_\lambda\right) = C_p(\lambda) \times \lambda^v. \tag{6.1.2}$$

Moreover, let h be the unique solution in $\left(0, \frac{1}{4}\right]$ of

$$\lambda = \frac{h}{(1+8h)^{3/2}}. \tag{6.1.3}$$

Then we have

$$C_p(\lambda) = \frac{1}{\lambda}\left(8+\frac{1}{h}\right)^{p-1}\sum_{q=0}^{p-1}\binom{2q}{q}h^q, \tag{6.1.4}$$

so the distribution of $\mathbb{T}_\lambda$ is completely explicit.

A very useful consequence of (6.1.2) is the spatial Markov property of $\mathbb{T}_\lambda$: for any triangulation t with a hole of perimeter p, conditionally on $t \subset \mathbb{T}_\lambda$, the distribution of the complement $\mathbb{T}_\lambda \backslash t$ only depends on p. Therefore, it is possible to discover it in a Markovian way by a *peeling exploration*.

Since this will be useful later, we recall basic definitions related to peeling explorations. A *peeling algorithm* $\mathcal{A}$ is a way to associate to every triangulation with holes an edge on the boundary of one of the holes. Given an infinite triangulation T and a peeling algorithm $\mathcal{A}$, we can define an increasing sequence $\left(\mathcal{E}_T^{\mathcal{A}}(k)\right)_{k \geqslant 0}$ of triangulations with holes such that $\mathcal{E}_T^{\mathcal{A}}(k) \subset T$ for every k in the following way:

- the map $\mathcal{E}_T^{\mathcal{A}}(0)$ is the trivial map consisting of the root edge only,
- for every $k \geqslant 1$, the triangulation $\mathcal{E}_T^{\mathcal{A}}(k+1)$ is obtained from $\mathcal{E}_T^{\mathcal{A}}(k)$ by adding the triangle incident to $\mathcal{A}\left(\mathcal{E}_T^{\mathcal{A}}(k)\right)$ outside of $\mathcal{E}_T^{\mathcal{A}}(k)$ and, if this triangle creates a finite hole, all the triangles in this hole.

Such an exploration is called *filled-in*, because all the finite holes are filled at each step. In particular, for the PSHT, we denote by $P^\lambda(k)$ and $V^\lambda(k)$ the perimeter and volume of $\mathcal{E}_T^{\mathcal{A}}(k)$. The spatial Markov property ensures that $\left(P^\lambda(k), V^\lambda(k)\right)_{k \geqslant 0}$ is a Markov chain on $\mathbb{N}^2$ and that its transitions do not depend on the algorithm $\mathcal{A}$. We also recall the asymptotic behaviour of these two processes. We have

$$\frac{P^\lambda(k)}{k} \xrightarrow[k\to+\infty]{a.s.} \sqrt{\frac{1-4h}{1+8h}} \qquad \text{and} \qquad \frac{V^\lambda(k)}{k} \xrightarrow[k\to+\infty]{a.s.} \frac{1}{\sqrt{(1+8h)(1-4h)}}, \tag{6.1.5}$$

where h is given by (6.1.3). These estimates are proved in [Cur16] in the type-II setting. For the type-I PSHT, the proofs are the same and use the combinatorial results of [Kri07]. In particular, the asymptotic ratio between $P^\lambda(k)$ and $V^\lambda(k)$ is $1-4h$, which is a decreasing function of λ. This shows that the PSHT for different values of λ are singular with respect to each other, which will be useful later.

6.1.4 Local convergence and dual local convergence

The goal of this section is to recall the definition of local convergence in a setting that is not restricted to planar maps. We also define a weaker (at least for triangulations) notion of local convergence that we call "dual local convergence".

As in the planar case, to define the local convergence, we first need to define balls of triangulations. Let t be a finite triangulation. As usual, for every $r \geqslant 1$, we denote by $B_r(t)$ the map formed by all the faces of t which are incident to at least one vertex at distance at most $r-1$ from the root vertex, along with all their vertices and edges. We denote by $\partial B_r(t)$ the set of edges e such that exactly one side of e is adjacent to a triangle of $B_r(t)$. The other sides of these edges form a finite number of holes, so $B_r(t)$ is a finite triangulation with holes. Note that contrary to the planar case, there is no bijection between the holes and the connected components of $t \backslash B_r(t)$ (cf. the component on the left of Figure 6.1). We also write $B_0(t)$ for the trivial "map" consisting of only one vertex and zero edge.

For any two finite triangulations t and t', we write

$$d_{\mathrm{loc}}(t,t') = \left(1+\max\{r \geqslant 0 | B_r(t)=B_r(t')\}\right)^{-1}.$$

This is the *local distance* on the set of finite triangulations. As in the planar case, its completion $\overline{\mathcal{T}}$ is a Polish space, which can be viewed as the set of (finite or infinite) triangulations in which all the vertices have finite degree. However, this space is not compact.

In some parts of this chapter, it will be more convenient to work with a weaker notion of convergence which we call the *dual local convergence*. The reason for this is that, since the degrees in the dual of a triangulation are bounded by 3, tightness for this distance will be immediate, which will allow us to work directly on infinite subsequential limits.

More precisely, we recall that d^* is the graph distance on the *dual* of a triangulation. For any finite triangulation t and any $r \geqslant 0$, we denote by $B_r^*(t)$ the map formed by all the faces at dual distance at most r from the root face, along with all their vertices and edges. Like $B_r(t)$, this is a finite triangulation with holes. For any two finite triangulations t and t', we write

$$d^*_{\mathrm{loc}}(t,t') = \left(1+\max\{r \geqslant 0 | B^*_r(t)=B^*_r(t')\}\right)^{-1}.$$

Note that in any triangulation t, since the dual graph of t is 3-regular, there are at most $3\times 2^{r-1}$ faces at distance r from the root face. Therefore, for each r, the volume of $B_r^*(t)$ is bounded by a constant depending only on r, so $B_r^*(t)$ can only take finitely many values. It follows that the completion for d^*_{loc} of the set of finite triangulations is compact. We write it $\overline{\mathcal{T}}^*$. This set coincides with the set of finite or infinite triangulations, where the degrees of the vertices may be infinite.

Roughly speaking, the main steps of our proof for tightness will be the following. Since $(\overline{\mathcal{T}}^*, d^*_{\mathrm{loc}})$ is compact, the sequence (T_{n,g_n}) is tight for d^*_{loc}. We will prove that every subsequential limit is planar and one-ended, and finally that its vertices must have finite degree. We state right now an easy, deterministic lemma that will allow us to conclude at this point.

Lemma 6.1.4. *Let (t_n) be a sequence of triangulations of $\overline{\mathcal{T}}$. Assume that*

$$t_n \xrightarrow[n\to+\infty]{d^*_{\mathrm{loc}}} t,$$

with $t \in \overline{\mathcal{T}}$. Then $t_n \to t$ for d_{loc} when $n \to +\infty$.

Note that the converse is very easy: the dual ball $B^*_r(t)$ is a deterministic function of $B_{r+1}(t)$, so $d^*_{\mathrm{loc}} \leqslant 2d_{\mathrm{loc}}$, and convergence for d_{loc} implies convergence for d^*_{loc}.

Proof of Lemma 6.1.4. Let $r \geqslant 1$. Since $t \in \overline{\mathcal{T}}$, the ball $B_r(t)$ is finite, so we can find r^* such that $B_r(t) \subset B^*_{r^*}(t)$. By definition of d^*_{loc}, for n large enough, we have $B^*_{r^*}(t_n) = B^*_{r^*}(t)$. Therefore, we have $B_r(t) \subset t_n$, so $B_r(t_n) = B_r(t)$ for n large enough. Since this is true for any $r \geqslant 1$, we are done. □

6.2 Tightness, planarity and one-endedness

6.2.1 The bounded ratio lemma

The goal of this section is to prove the following result, which will be our main new input in the proof of tightness.

Lemma 6.2.1 (Bounded ratio lemma)**.** *Let $\varepsilon > 0$. Then there is a constant $C_\varepsilon > 0$ with the following property: for every $n, g \geqslant 0$ satisfying $\frac{g}{n} \leqslant \frac{1}{2} - \varepsilon$ and for every $p \geqslant 1$, we have*

$$\frac{\tau_p(n,g)}{\tau_p(n-1,g)} \leqslant C_\varepsilon.$$

In particular, by the usual bijection of Figure 6.2 between $\mathcal{T}_1(n,g)$ and $\mathcal{T}(n,g)$, we have

$$\frac{\tau(n,g)}{\tau(n-1,g)} \leqslant C_\varepsilon.$$

For our future use, it will be important that the constant C_ε does not depend on p. The idea of the proof of Lemma 6.2.1 will be to find an "almost-injective" way to obtain a triangulation of $\mathcal{T}_p(n-1,g)$ from a triangulation of $\mathcal{T}_p(n,g)$. This will be done by merging two vertices together. For this, it will be useful to find two vertices that are quite close from each other and have a reasonable (i.e. bounded) degree. This is the point of the next result. We recall that for two vertices v, v', the distance $d^*(v,v')$ is the length of the smallest dual path from v to v' that avoids the external faces.

Lemma 6.2.2. *In any triangulation of a polygon t with parameters n and g satisfying the conditions of Lemma 6.2.1, there are at least $\frac{\varepsilon}{12}n$ pairs of vertices (v_1, v_2) such that*

$$\deg(v_1) + \deg(v_2) \leqslant \frac{12}{\varepsilon}$$

and $d^*(v_1,v_2) \leqslant \frac{24}{\varepsilon}$.

Proof. Fix a triangulation $t \in \mathcal{T}_p(n,g)$ with (n,g) satisfying the conditions of Lemma 6.2.1. Pairs of vertices satisfying the conclusion will be called *good pairs*. We first note that a positive proportion of the vertices have a small degree. Indeed, by the Euler formula, t has $3n+3-p$ edges and $n+2-2g$ vertices, so the average degree of a vertex is

$$\frac{2(3n+3-p)}{n+2-2g} \leqslant \frac{6n}{2\varepsilon n} = \frac{3}{\varepsilon}.$$

Therefore, at least half of the vertices have degree at most $\frac{6}{\varepsilon}$. There are $n+2-2g \geqslant 2\varepsilon n$ vertices in t, so at least εn of them have degree not greater than $\frac{6}{\varepsilon}$.

For each of these εn vertices v, fix a triangular face f_v incident to v. Since each face is incident to only 3 vertices, the set F of the faces f_v contains at least $\frac{\varepsilon}{3}n$ faces. We need to find $\frac{\varepsilon}{12}n$ pairs $(f,f') \in F^2$ with $d^*(f,f') \leqslant \frac{24}{\varepsilon}$. This will follow from the fact that balls (for d^*) centered at the elements of F must strongly overlap. More precisely, let $r = \frac{12}{\varepsilon}$. Since the dual map t^* is connected, for any $f \in F$, we have[4] $|B_r^*(f)| \geqslant r$. Therefore, we have

$$\sum_{f \in F} |B_r^*(f)| \geqslant \frac{12}{\varepsilon} \times \frac{\varepsilon}{3} n = 4n,$$

[4]Unless the number of faces of t is smaller than r, in which case $B_r^*(f) = t^*$, so any pair of F^2 is good.

whereas $|t^*| = 2n-p+2 < 4n$. Hence, there must be an intersection between the balls $B_r^*(f)$, so there are $f_1, f_1' \in F$ such that $d^*(f_1, f_1') \leqslant 2r = \frac{24}{\varepsilon}$, and the pair (f_1, f_1') is good. We set $F_1 = F \backslash \{f_1\}$. We now try to find a good pair in F_1 and remove an element of this pair, and so on. Assume that F_i is the set F where i elements have been removed. If $i < \frac{\varepsilon}{12}n$, then we have

$$\sum_{f \in F_i} |B_r^*(f)| \geqslant \frac{12}{\varepsilon}\left(\frac{\varepsilon}{3}n - i\right) \geqslant 3n > |t^*|,$$

so F_i contains a good pair. Therefore, the process will not stop before $i = \frac{\varepsilon}{12}n$, so we can find $\frac{\varepsilon}{12}n$ good pairs in F^2, which concludes the proof. $\square$

We are now able to prove the bounded ratio lemma.

Proof of Lemma 6.2.1. Let $g \geqslant 0, n \geqslant 2$ be such that $\frac{g}{n} \leqslant \frac{1}{2} - \varepsilon$. We will define an "almost-injection" Φ from $\mathcal{T}_p(n,g)$ to $\mathcal{T}_p(n-1,g)$. The input will be a triangulation $t \in \mathcal{T}(n,g)$ with a marked good pair (v_1, v_2). By Lemma 6.2.2, the number of inputs is at least

$$\frac{\varepsilon}{12} n \tau_p(n,g). \tag{6.2.1}$$

Given t, v_1 and v_2, let $(f_1, f_2, \ldots, f_j)$ be the shortest path in t^* between v_1 and v_2. Since the pair (v_1, v_2) is good, we have $j \leqslant \frac{24}{\varepsilon}$. For all i, let e_i be the edge separating f_i and f_{i+1}. We flip e_1, then e_2 and so on up to e_{j-1} (see Figure 6.3). Note that these flips are always well-defined, since the faces f_i are pairwise distinct.

As we do so, we keep track of all the edges flipped and the order they come in. All the edges that were flipped are now incident to v_1, and the last of them is also incident to v_2. We then contract this edge and merge v_1 and v_2 into a vertex v, which creates two digons incident to v, that we contract into two edges. Finally, we mark the vertex v obtained by merging v_1 and v_2, and we also mark the two edges obtained by contracting the digons.

These operations do not change the genus and the boundary length, and remove exactly 1 vertex. Hence, the output of Φ is a triangulation of $\mathcal{T}_p(n-1,g)$, with a marked vertex v of degree at most $\frac{36}{\varepsilon}$ (this is because $\deg(v) < \deg(v_1) + \deg(v_2) + j$), two marked edges incident to v and an ordered list of edges incident to v. Moreover, given the triangulation $\Phi(t)$, there are at most $n+2-2g \leqslant n+2$ possible values of v. Since $\deg(v) \leqslant \frac{36}{\varepsilon}$, once v is fixed, there are at most $\left(\frac{36}{\varepsilon}\right)^2$ ways to choose the two marked edges and

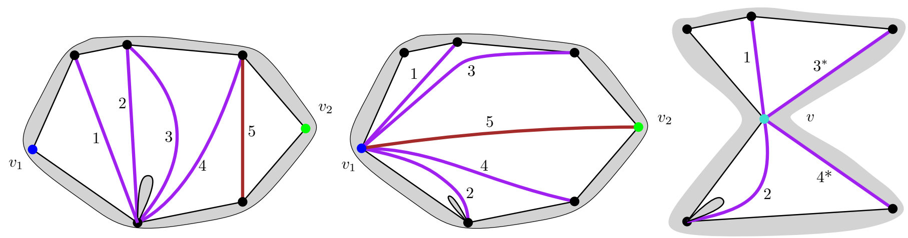


Figure 6.3 – The injection. On the left, a good pair and a path of triangles. In the center, the triangulation after the flips (we flipped the edges $1, 2, 3, 4, 5$ in this order). On the right, the final map, after contraction of the brown edge. The stars indicate the contracted digons.

$\left(\frac{36}{\varepsilon}\right)^{24/\varepsilon}$ to choose the ordered list of edges. Hence, the number of possible outputs of Φ is at most

$$n\left(\frac{36}{\varepsilon}\right)^{24/\varepsilon+2}\tau_p(n-1,g). \tag{6.2.2}$$

Finally, it is easy to see that Φ is injective: to go backwards, one just needs to duplicate the two marked edges, split v in two between the two digons, and flip back the edges in the prescribed order. Therefore, by (6.2.1) and (6.2.2), we obtain

$$\frac{\varepsilon}{12}n\tau_p(n,g) \leqslant n\left(\frac{36}{\varepsilon}\right)^{24/\varepsilon+2}\tau_p(n-1,g),$$

which concludes the proof with $C_\varepsilon = \left(\frac{36}{\varepsilon}\right)^{24/\varepsilon+3}$. □

6.2.2 Planarity and one-endedness

We now fix a sequence (g_n) with $\frac{g_n}{n} \to \theta \in \left[0, \frac{1}{2}\right)$. As explained in Section 6.1.4, the tightness of (T_{n,g_n}) for d^*_{loc} is immediate. In all this section, we will denote by T a subsequential limit in distribution. It must be an infinite triangulation. We will first prove that T is planar and one-ended, and then that its vertices have finite degrees. To establish planarity, the idea will be to bound, for any non-planar finite triangulation t, the probability that $t \subset T_{n,g_n}$ for n large. For this, we will need the following combinatorial estimate.

Lemma 6.2.3. *Fix $k \geqslant 1$ and $m \in \mathbb{Z}$, numbers $\ell_1, \dots \ell_k \geqslant 1$ and perimeters $p_i^j \geqslant 1$ for $1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant \ell_j$. Then*

$$\sum_{\substack{n_1+\dots+n_k=n+m \\ h_1+\dots+h_k=g_n-1-\sum_j(\ell_j-1)}} \prod_{j=1}^{k} \tau_{p_1^j,\dots,p_{\ell_j}^j}(n_j,h_j) = o\left(\tau(n,g_n)\right) \tag{6.2.3}$$

when $n \to +\infty$.

Proof. By Lemma 6.1.2, the left-hand side of (6.2.3) can be bounded by

$$C \sum_{\substack{n_1+\dots+n_k=n+m \\ h_1+\dots+h_k=g_n-1-\sum_j(\ell_j-1)}} \prod_{j=1}^{k} n_j^{\ell_j-1}\tau(n_j,h_j)$$

for $C = 6^{\sum(\ell_j-1)}$. We now use the crude bound $\tau(n_j,h_j) \leqslant f(n_j,h_j)$, where the numbers $f(n,g) = (3n+2)\tau(n,g)$ are those that appear in the Goulden–Jackson formula (6.1.1). Since $\ell_j \geqslant 1$, we can bound $n_j^{\ell_j-1}$ by $2n^{\ell_j-1}$ for n large enough. The left-hand side of (6.2.3) is then bounded by

$$Cn^{\sum_{j=1}^{k}(\ell_j-1)} \sum_{\substack{n_1+\dots+n_k=n+m \\ h_1+\dots+h_k=g_n-1-\sum_j(\ell_j-1)}} \prod_{j=1}^{k} f(n_j,h_j).$$

Moreover, the Goulden–Jackson formula implies that

$$\sum_{\substack{n_1+n_2=n \\ h_1+h_2=g}} f(n_1,h_1)f(n_2,h_2) \leqslant f(n+2,g)$$

for any $n, g \geqslant 0$. By an easy induction on k, we obtain

$$\sum_{\substack{n_1+\dots+n_k=n \\ h_1+\dots+h_k=g}} \prod_{j=1}^{k} f(n_j,h_j) \leqslant f(n+2k-2,g).$$

Therefore, we can bound the left-hand side of (6.2.3) by

$$Cn^{\sum_{j=1}^{k}(\ell_j-1)} f\left(n+m+2k-2, g_n-1-\sum_{j=1}^{k}(\ell_j-1)\right). \tag{6.2.4}$$

The Goulden–Jackson formula implies $f(n-2,g-1) \leqslant n^{-2}f(n,g)$ for any n and g, so $f(n,g-i) \leqslant n^{-2i}f(n+2i,g)$ for any n and $1 \leqslant i \leqslant g$.

Therefore, from (6.2.4), we obtain the bound

$$Cn^{\sum_{j=1}^k(\ell_j-1)}n^{-2-2\sum_{j=1}^k(\ell_j-1)}f\left(n+m+2k+2\sum(\ell_j-1),g_n\right)$$
$$\leqslant C'n^{-2-\sum_j(\ell_j-1)}f(n,g_n),$$

where in the end we use Lemma 6.2.1 (which results in a change in the constant). In particular, the left-hand side of (6.2.3) is $o\left(\frac{f(n,g_n)}{n}\right)$, so it is $o\left(\tau(n,g_n)\right)$. $\square$

Corollary 6.2.4. *Every subsequential limit of (T_{n,g_n}) for d^*_{loc} is a.s. planar.*

Proof. If a subsequential limit T is not planar, then we can find a finite triangulation t with holes and with genus 1 such that $t \subset T$. Indeed, if we explore T triangle by triangle, the genus may only increase by at most 1 at each step, so if the genus is positive at some point, it must be 1 at some point. Therefore, it is enough to prove that for any such triangulation t, we have

$$\mathbb{P}\left(t \subset T_{n,g_n}\right) \xrightarrow[n\to+\infty]{} 0.$$

If $t \subset T_{n,g_n}$, let $T^1,\dots,T^k$ be the connected components of $T_{n,g_n}\backslash t$. These components define a partition of the set of holes of t, where a hole h is in the j-th class if T^j is the connected component glued to h (for example, on Figure 6.1, the three classes have sizes 2, 1 and 1). Note that the number of possible partitions is finite and depends only on t (and not on n). Therefore, it is enough to prove that for any partition π of the set of holes of t, we have

$$\mathbb{P}\left(t \subset T_{n,g_n} \text{ and the partition defined by } T_{n,g_n} \text{ is } \pi\right) \xrightarrow[n\to+\infty]{} 0. \qquad (6.2.5)$$

If this occurs, for each j, let ℓ_j be the number of holes of T glued to T^j and let $p^j_1,\dots,p^j_{\ell_j}$ be the perimeters of these holes. Then the connected component T^j is a triangulation of the $(p^j_1,\dots,p^j_{\ell_j})$-gon (see Figure 6.1). Moreover, if T_j has genus h_j, then the total genus of T_{n,g_n} is equal to

$$1+\sum_{j=1}^k h_j+\sum_{j=1}^k(\ell_j-1),$$

so this sum must be equal to g_n, so

$$\sum_{j=1}^k h_j = g_n-1-\sum_{j=1}^k(\ell_j-1).$$

Moreover, let n_j be such that T^j belongs to $\mathcal{T}_{p_1^j,\dots,p_{\ell_j}^j}(n_j,h_j)$. An easy computation shows that

$$\sum_{j=1}^{k} n_j = n+m$$

with

$$m = \frac{1}{2}\left(-|F(t)| + \sum_{j=1}^{k}\sum_{i=1}^{\ell_j}(p_i^j-2)\right) \in \mathbb{Z},$$

where $|F(t)|$ is the number of triangles of t.

Therefore, the number of triangulations $T \in \mathcal{T}(n,g_n)$ such that $t \subset T$ and the resulting partition of the holes is equal to π is the number of ways to choose, for each j, a triangulation of the $(p_1^j,\dots,p_{\ell_j}^j)$-gon, such that the total genus of these triangulations is $g_n - 1 - \sum_{j=1}^{k}(\ell_j-1)$, and their total size is $n+m$. This is equal to the left-hand side of Lemma 6.2.3, so (6.2.5) is a consequence of Lemma 6.2.3, which concludes the proof. $\square$

The proof of one-endedness will be similar, but the combinatorial estimate that is needed is slightly different.

Lemma 6.2.5.
- *Fix $k \geqslant 1$, $m \in \mathbb{Z}$, numbers $\ell_1,\dots\ell_k \geqslant 1$ that are not all equal to 1, and perimeters $p_i^j \geqslant 1$ for $1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant \ell_j$. Then*

$$\sum_{\substack{n_1+\dots+n_k=n+m\\ h_1+\dots+h_k=g_n-\sum_j(\ell_j-1)}} \prod_{j=1}^{k} \tau_{p_1^j,\dots,p_{\ell_j}^j}(n_j,h_j) = o\left(\tau(n,g_n)\right). \tag{6.2.6}$$

- *Fix $k \geqslant 2$, $m \in \mathbb{Z}$ and perimeters $p_1,\dots,p_k$. There is a constant C such that, for every a and n, we have*

$$\sum_{\substack{n_1+\dots+n_k=n+m\\ h_1+\dots+h_k=g_n\\ n_1,n_2>a}} \prod_{j=1}^{k} \tau_{p_j}(n_j,h_j) \leqslant \frac{C}{a}\tau(n,g_n). \tag{6.2.7}$$

Proof. We start with the first point. The proof is very similar to the proof of Lemma 6.2.3, with the following difference: here, the sum of the genuses differs by one, so we will lose a factor n^2 in the end of the computation. This forces us to be more careful in the beginning and to use the assumption that the ℓ_j are not all equal to 1.

More precisely, by using Lemma 6.1.2 and the bound $\tau(n,g) \leqslant \frac{1}{n} f(n,g)$, as in the proof of Lemma 6.2.3, the left-hand side of (6.2.6) can be bounded by

$$C \sum_{\substack{n_1+\dots+n_k=n+m \\ h_1+\dots+h_k=g_n-\sum_j(\ell_j-1)}} \left(\prod_{j=1}^{k} n_j^{\ell_j-2}\right)\left(\prod_{j=1}^{k} f(n_j,h_j)\right),$$

where C does not depend on n. Without loss of generality, assume that $\ell_1 \geqslant 2$. Then we have $n_1^{\ell_1-2} \leqslant (n+m)^{\ell_1-2}$. Moreover, for every $j \geqslant 2$, we have $n_j^{\ell_j-2} \leqslant n_j^{\ell_j-1} \leqslant (n+m)^{\ell_j-1}$ since $\ell_j \geqslant 1$. Therefore, we obtain for n large enough

$$\prod_{j=1}^{k} n_j^{\ell_j-2} \leqslant 2n^{\sum_j(\ell_j-1)-1}.$$

By using this and the Goulden–Jackson formula in the same way as in the proof of Lemma 6.2.3, we obtain the bound

$$Cn^{-1-\sum_j(\ell_j-1)} f\left(n+m+2k-2, g_n\right) \leqslant C' n^{-1-\sum_j(\ell_j-1)} f(n,g_n),$$

where the last inequality follows from the bounded ratio lemma. Since $\ell_1 \geqslant 2$, we have $\sum_j(\ell_j-1) \geqslant 1$, so this is $o\left(\frac{f(n,g_n)}{n}\right)$ and we get the result.

We now prove the second point. As in the first case (but with $\ell_j = 1$ for every j), the left-hand side can be bounded by

$$C \sum_{\substack{n_1+\dots+n_k=n+m \\ h_1+\dots+h_k=g_n \\ n_1,n_2>a}} \left(\prod_{j=1}^{k} \frac{1}{n_j}\right)\left(\prod_{j=1}^{k} f(n_j,h_j)\right).$$

Moreover, if $n_1, n_2 > a$, then at least one of the n_j is larger than $\frac{n+m}{k}$ and two are larger than a, so $\prod_{j=1}^k n_j \geqslant \frac{(n+m)a}{k}$, so we obtain the bound (for n large enough, with C' and C'' independent of n and a)

$$\begin{aligned}
\frac{C'}{an} \sum_{\substack{n_1+\dots+n_k=n+m \\ h_1+\dots+h_k=g_n \\ n_1,n_2>a}} \prod_{j=1}^{k} f(n_j,h_j) &\leqslant \frac{C'}{an} \sum_{\substack{n_1+\dots+n_k=n+m \\ h_1+\dots+h_k=g_n}} \prod_{j=1}^{k} f(n_j,h_j) \\
&\leqslant \frac{C'}{an} f(n+m+2k-2, g_n) \\
&\leqslant \frac{C''}{an} f(n,g_n) \\
&\leqslant \frac{C''}{a} \tau(n,g_n),
\end{aligned}$$

where we use the Goulden–Jackson formula to reduce the sum and finally the bounded ratio lemma, in the same way as previously. $\square$

Corollary 6.2.6. *Every subsequential limit of (T_{n,g_n}) for d^*_{loc} is a.s. one-ended.*

Proof. The proof is quite similar to the proof of Corollary 6.2.4, but with Lemma 6.2.5 playing the role of Lemma 6.2.3.

More precisely, if a subsequential limit T is not one-ended with positive probability, it contains a finite triangulation t such that two of the connected components of $T\backslash t$ are infinite. This means that we can find $\varepsilon > 0$, a triangulation t and two holes h_1, h_2 of t such that, for every $a > 0$,

$$\mathbb{P}\left(t \subset T \text{ and } T\backslash t \text{ has two connected components with at least } a \text{ faces}\right) \geqslant \varepsilon. \tag{6.2.8}$$

By Corollary 6.2.4, we can assume that t is planar. If this holds, then T contains a finite triangulation obtained by starting from t and adding a faces in the hole h_1 and a faces in the hole h_2. We denote by $t^{a,a}$ the set of such triangulations. Then (6.2.8) means that for any $a > 0$, for n large enough, we have

$$\mathbb{P}\left(T_{n,g_n} \text{ contains a triangulation of } t^{a,a}\right) \geqslant \varepsilon. \tag{6.2.9}$$

This can occur in two different ways, which will correspond to the two items of Lemma 6.2.5:

(i) either at least one connected component of $T_{n,g_n}\backslash t$ is adjacent to at least two holes of t,

(ii) or the k holes of t correspond to k connected components $T^1, \dots, T^k$, where T^1 and T^2 have size at least a.

In case (i), the connected components of T_{n,g_n} are triangulations of multi-polygons, at least one of which has two boundaries. The proof that the probability of this case goes to 0 is now the same as the proof of Corollary 6.2.4, but we use the first point of Lemma 6.2.5. Note that the assumption the ℓ_j are not all 1 comes from the fact that one of the connected components is adjacent to two holes. Moreover, the sum of the genuses of the T^j is $g - \sum_j(\ell_j - 1)$ and not $g - 1 - \sum_j(\ell_j - 1)$ because this time t has genus 0 and not 1.

Similarly, in case (ii), the k holes of t must be filled with k triangulations of a single polygon, two of which have at least a faces, so they belong to a set of the form $\mathcal{T}_{p_j}(n_j, h_j)$ with $n_j \geqslant \frac{a}{2}$ if a is large enough compared to the perimeters of the holes. Hence, the second point of Lemma 6.2.5 allows to bound the number of ways to fill these holes. We obtain that, for a large enough, we have

$$\mathbb{P}\left(T_{n,g_n} \text{ contains a triangulation of } t^{a,a}\right) \leqslant o(1) + \frac{2C}{a}$$

as $n \to +\infty$, where $o(1)$ comes from case (i) and $\frac{2C}{a}$ from case (ii). This contradicts (6.2.9), so T is a.s. one-ended. □

6.2.3 Finiteness of the degrees

Our goal is now to prove tightness for d_{loc}. As before, let (g_n) be a sequence with $\frac{g_n}{n} \to \theta \in \left[0, \frac{1}{2}\right)$.

Proposition 6.2.7. *The sequence (T_{n,g_n}) is tight for d_{loc}.*

Let T be a subsequential limit of (T_{n,g_n}) for d^*_{loc}. By Lemma 6.1.4, to finish the proof of tightness for d_{loc}, we only need to show that almost surely, all the vertices of T have finite degree. As in [AS03], we will first study the degree of the root vertex, and then extend finiteness by using invariance under the simple random walk. The main difference with [AS03] is that, while [AS03] uses exact enumeration results, we will rely on the bounded ratio lemma.

Lemma 6.2.8. *The root vertex of T has a.s. finite degree.*

Proof. We follow the approach of [AS03] and perform a filled-in peeling exploration of T. Before specifying the peeling algorithm that we use, note that we already know by Corollary 6.2.4 that the explored part will always be planar, so no peeling step will merge two different existing holes. Moreover, by Corollary 6.2.6, if a peeling step separates the boundary into two holes, then one of them is finite and will be filled with a finite triangulation. Therefore, at each step, the explored part will be a triangulation with a single hole.

The peeling algorithm $\mathcal{A}$ that we use is the following: if the root vertex ρ belongs to ∂t, then $\mathcal{A}(t)$ is the edge on ∂t on the left of ρ. If $\rho \notin \partial t$, then the exploration is stopped. Since only finitely many edges incident to ρ are added at each step, it is enough to prove that the exploration will a.s. eventually stop. We recall that $\mathcal{E}^{\mathcal{A}}_T(i)$ is the explored part at time i.

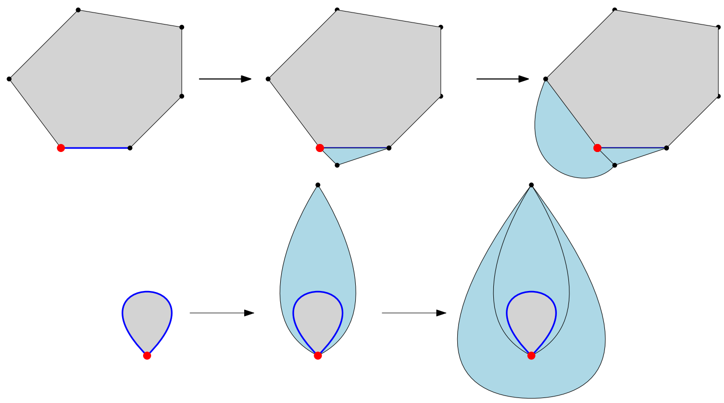


Figure 6.4 – The construction of t^+ from t. In gray, the triangulation t. In red, the root vertex. In blue, the new triangles. In the bottom, the case $|\partial t| = 1$.

We will prove that at each step, conditionally on $\mathcal{E}_T^{\mathcal{A}}(i)$, the probability to swallow the root and finish the exploration at time $i+1$ or $i+2$ is bounded from below by a positive constant. For every triangulation t with one hole such that $\rho \in \partial t$, we denote by t^+ the triangulation constructed from t as follows (see Figure 6.4):

- we first glue a triangle to the edge of ∂t on the left of ρ, in such a way that the third vertex of this triangle does not belong to t, to obtain a triangulation with perimeter at least 2;
- we then glue a second triangle to the two edges of the boundary incident to ρ.

Note that t^+ is a planar map with the same perimeter as t but one more vertex and two more triangles. By the choice of our peeling algorithm, if $\mathcal{E}_T^{\mathcal{A}}(i)^+ \subset T$, then we have $\mathcal{E}_T^{\mathcal{A}}(i+2) = \mathcal{E}_T^{\mathcal{A}}(i)^+$. Moreover, if this is the case, the exploration is stopped at time $i+2$. Hence, it is enough to prove that the quantity

$$\mathbb{P}\left(t^+ \subset T | t \subset T\right)$$

is bounded from below for finite, planar triangulations t with a single hole and $\rho \in \partial t$.

We fix such a t, with perimeter p. Let also (n_k) be a sequence of indices such that $T_{n_k,g_{n_k}}$ converges in distribution to T. We have

$$\mathbb{P}\left(t^+ \subset T | t \subset T\right) = \lim_{k\to+\infty} \frac{\mathbb{P}\left(t^+ \in T_{n_k,g_{n_k}}\right)}{\mathbb{P}\left(t \in T_{n_k,g_{n_k}}\right)} = \lim_{k\to+\infty} \frac{\tau_p\left(n_k+m-1, g_{n_k}\right)}{\tau_p(n_k+m, g_{n_k})},$$

where $m = \frac{p-2-|F(t)|}{2}$ and $|F(t)|$ is the number of triangles of t. Moreover, there is $\varepsilon > 0$ such that $g_n \leqslant \left(\frac{1}{2} - 2\varepsilon\right) n$ for n large enough, so $\frac{g_{n_k}}{n_k+m} \leqslant \frac{1}{2} - \varepsilon$ for k large enough. By the bounded ratio lemma, we obtain

$$\mathbb{P}\left(t^+ \subset T | t \subset T\right) \geqslant \frac{1}{C_\varepsilon}$$

for every t, which concludes the proof. $\square$

Remark 6.2.9. *Our proof shows that the number of steps needed to swallow the root has exponential tail. However, since we do not control the finite triangulations filling the holes that may appear, it does not give any quantitative bound on the root degree.*

Proof of Proposition 6.2.7. Let T be a subsequential limit of (T_{n,g_n}). Because of Lemma 6.1.4, it is enough to prove that almost surely, all the vertices of T have finite degrees. The argument is essentially the same as in [AS03] and relies on Lemma 6.2.8 and invariance under the simple random walk.

More precisely, for every n, let $\overrightarrow{e_0}^n$ be the root edge of T_{n,g_n} and let $\overrightarrow{e_0}$ be the root of T. We first note that the distribution of T_{n,g_n} is invariant under reversing the orientation of the root, so this is also the case of T. By Lemma 6.2.8, this implies that the endpoint of $\overrightarrow{e_0}$ has a.s. finite degree.

We then denote by $\overrightarrow{e_1}^n$ the first step of the simple random walk on T_{n,g_n}: its starting point is the endpoint of $\overrightarrow{e_0}^n$ and its endpoint is picked uniformly among all the neighbours of the starting point. Since the endpoint of $\overrightarrow{e_0}$ has finite degree, we can also define the first step $\overrightarrow{e_1}$ of the simple random walk on T. For the same reason as in the planar case (see Theorem 3.2 of [AS03]), the triangulations $(T_{n,g_n}, \overrightarrow{e_1}^n)$ and $(T_{n,g_n}, \overrightarrow{e_0}^n)$ have the same distribution, so $(T, \overrightarrow{e_1})$ has the same distribution as $(T, \overrightarrow{e_0})$. In particular, all the neighbours of the endpoint of $\overrightarrow{e_0}$ must have finite degrees. From here, an easy induction on i shows that for every $i \geqslant 0$, we can define the i-th step $\overrightarrow{e_i}$ of the simple random walk on T, that $(T, \overrightarrow{e_i})$ has the same distribution as $(T, \overrightarrow{e_0})$ and that all vertices at distance i from the root in T are finite. This concludes the proof. $\square$

6.3 Weakly Markovian triangulations

The goal of this section is to prove Theorem 6.0.2. We first recall the definition of a weakly Markovian triangulation.

Definition 6.3.1. Let T be a random infinite triangulation of the plane. We say that T is *weakly Markovian* if there is a family $(a_v^p)_{v \geqslant p \geqslant 1}$ of numbers with the following property: for every triangulation t with a hole of perimeter p and v vertices in total, we have

$$\mathbb{P}\left(t \subset T\right) = a_v^p.$$

By their definition, the PSHT $\mathbb{T}_\lambda$ are weakly Markovian. We denote by $a_v^p(\lambda) = C_p(\lambda)\lambda^v$ the associated constants, where $C_p(\lambda)$ is given by (6.1.4). This implies that any mixture of these is also weakly Markovian. Indeed, for any random variable Λ with values in $(0, \lambda_c]$, we denote by $\mathbb{T}_\Lambda$ the PSHT with random parameter Λ. Let also μ be the distribution of Λ. Then for every triangulation t with a hole of perimeter p and v vertices in total, we have

$$\mathbb{P}\left(t \subset \mathbb{T}_\Lambda\right) = \int_0^{\lambda_c} \mathbb{P}\left(t \subset \mathbb{T}_\lambda\right) \mu(\mathrm{d}\lambda) = \int_0^{\lambda_c} C_p(\lambda)\lambda^v \mu(\mathrm{d}\lambda) =: a_v^p[\mu]. \tag{6.3.1}$$

Note that the last integral always converges since $C_p(\lambda)\lambda^v$ is bounded by 1. Therefore, the triangulation $\mathbb{T}_\Lambda$ is weakly Markovian. Our goal here is to prove Theorem 6.0.2, which states that the converse is true.

In all the rest of this section, we will denote by T some weakly Markovian random triangulation, and by $(a_v^p)_{v \geqslant p \geqslant 1}$ the associated constants. Before giving an idea of the proof, let us start with a remark that will be very useful in all that follows. The numbers a_v^p are linked to each other by linear equations that we call the *peeling equations*. In this section, for every $p \geqslant 1$ and $j \geqslant 0$, we denote by $|\mathcal{T}_p(j)|$ the number of planar triangulations of a p-gon (rooted on the boundary) with exactly j inner vertices[5].

Lemma 6.3.1. *For every $v \geqslant p \geqslant 1$, we have*

$$a_v^p = a_{v+1}^{p+1} + 2\sum_{i=0}^{p-1}\sum_{j=0}^{+\infty} |\mathcal{T}_{i+1}(j)| a_{v+j}^{p-i}. \tag{6.3.2}$$

[5]In order to have nicer formulas, the convention we use here differs from the rest of the chapter, in which the parameter n is related to the *total* number of vertices. We insist that this holds only in Section 6.3.

In particular, the sum in the right-hand side must converge. Note also that if $v \geqslant p \geqslant 1$, then $v + j \geqslant p - i \geqslant 1$ in all the terms of the sum, so all the terms make sense.

Proof of Lemma 6.3.1. The proof just consists of making one peeling step. Assume that $t \subset T$ for some triangulation t with a hole of perimeter p and volume v. Fix an edge $e \in \partial T$, and consider the face f out of t that is adjacent to e. Then we are in exactly one of the three following cases:

- the third vertex of f does not belong to ∂t,
- the third vertex of f belongs to ∂t, and f separates on its left i edges of ∂t from infinity,
- the third vertex of f belongs to ∂t, and f separates on its right i edges of ∂t from infinity.

In the first case, the triangulation we obtain by adding f to t has perimeter $p + 1$ and $v + 1$ vertices in total. In the other two cases, f separates $T \backslash t$ in one finite and one infinite components. The finite component has perimeter $i + 1$. If it has j inner vertices, then after filling the finite component, we obtain a triangulation with perimeter $p - i$ and volume $v + j$. $\square$

Our main job will be to prove the following result: it states that the numbers a_v^1 for $v \geqslant 1$ are compatible with some mixture of PSHT.

Proposition 6.3.2. *There is a probability measure μ on $(0, \lambda_c]$ such that, for every $v \geqslant 1$, we have*

$$a_v^1 = a_v^1[\mu].$$

Once Proposition 6.3.2 is proved, our main theorem follows easily. Indeed, Lemma 6.3.1 can be rewritten

$$a_{v+1}^{p+1} = a_v^p - 2 \sum_{i=0}^{p-1} \sum_{j=0}^{+\infty} |\mathcal{T}_{i+1}(j)| a_{v+j}^{p-i} \tag{6.3.3}$$

for $v + 1 \geqslant p + 1 \geqslant 2$. Hence, numbers of the form a_v^{p+1} can be expressed in terms of the numbers a_v^i with $i \leqslant p$. Therefore, by induction on p, we can prove that for every $v \geqslant p \geqslant 1$, we have

$$a_v^p = a_v^p[\mu].$$

Since the numbers a_v^p characterize entirely the distribution of T, we are done.

Therefore, we only need to prove Proposition 6.3.2. As a particular case of (6.3.1), for every measure μ and any $v \geqslant 0$, we have

$$a^1_{v+1}[\mu] = \int_0^{\lambda_c} C_1(\lambda)\lambda^{v+1}\mu(\mathrm{d}\lambda) = \int_0^{\lambda_c} \lambda^v \mu(\mathrm{d}\lambda).$$

The right-hand side can be interpreted as the v-th moment of a random variable with distribution μ. Therefore, all we need to prove is that there is a random variable Λ with support in $(0, \lambda_c]$ such that, for every $v \geqslant 0$, we have

$$a^1_{v+1} = \mathbb{E}[\Lambda^v].$$

We first show that there exists such a variable with support in $[0, 1]$. Since $a^1_1 = 1$, this is precisely the Hausdorff moment problem, so all we need to show is that the sequence $(a^1_{v+1})_{v \geqslant 0}$ is completely monotonic. More precisely, let Δ be the discrete derivative operator:

$$(\Delta u)_n = u_n - u_{n+1}.$$

Then, to prove that $(a^1_{v+1})_{v \geqslant 0}$ is the moment sequence of a probability measure on $[0, 1]$, it is sufficient to prove the following lemma.

Lemma 6.3.3. *For every $k \geqslant 0$ and $v \geqslant p \geqslant 1$, we have*

$$\left(\Delta^k a^p\right)_v \geqslant 0.$$

Note that only the case $p = 1$ is necessary. However, it will be more convenient to prove the lemma in the general case.

Proof. We prove the lemma by induction on k. The case $k = 0$ just means that $a^p_v \geqslant 0$ for every $v \geqslant p \geqslant 1$. The case $k = 1$ means that a^p_v is non-decreasing in v, which is a straightforward consequence of the peeling equation: in the right-hand side of (6.3.2), the term corresponding to $i = 0$ and $j = 1$ is $|\mathcal{T}_1(1)| a^p_{v+1}$, so

$$a^p_v \geqslant 2a^p_{v+1} \geqslant a^p_{v+1}.$$

Now assume that $k \geqslant 1$ and that the lemma is proved for k. We will use the result for k and $p+1$ to prove it for $k+1$ and p. Let $v \geqslant p \geqslant 1$. By using the induction hypothesis and (6.3.3) for $(p, v), (p, v+1), \dots, (p, v+k)$, we obtain

$$
\begin{aligned}
0 &\leqslant \Delta^k (a^{p+1})_{v+1} \\
&= \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} a^{p+1}_{v+1+\ell} \\
&= \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} a^{p}_{v+\ell} - 2 \sum_{i=0}^{p-1} \sum_{j=0}^{+\infty} |\mathcal{T}_{i+1}(j)| \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} a^{p-i}_{v+\ell+j} \\
&= \left(\Delta^k a^p\right)_v - 2 \sum_{i=0}^{p-1} \sum_{j=0}^{+\infty} |\mathcal{T}_{i+1}(j)| \left(\Delta^k a^{p-i}\right)_{v+j}.
\end{aligned}
$$

By the induction hypothesis, all the terms $\left(\Delta^k a^{p-i}\right)_{v+j}$ in the sum are non-negative. Therefore, the above sum does not decrease if we remove some of the terms. In particular, we may remove all the terms except the one for which $i = 0$ and $j = 1$, and we may also remove the factor 2. Since $|\mathcal{T}_1(1)| = 1$, we obtain

$$
0 \leqslant \left(\Delta^k a^p\right)_v - \left(\Delta^k a^p\right)_{v+1} = \left(\Delta^{k+1} a^p\right)_v,
$$

which proves the lemma by induction. □

Remark 6.3.4. *Many bounds used in the last proof may seem very crude. This is due to the fact that we prove the existence of a variable Λ with support in $[0,1]$, whereas its support is actually in $[0,\lambda_c]$. For example, if we had not got rid of the factors 2, we would have obtained that Λ has support in $\left[0,\frac{1}{2}\right]$ instead of $[0,1]$.*

End of the proof of Theorem 6.0.2. By Lemma 6.3.3, there is a random variable Λ with support in $[0,1]$ such that, for every $v \geqslant 1$, we have $a^1_{v+1} = \mathbb{E}[\Lambda^v]$. All we need to show is that $\Lambda \in (0,\lambda_c]$ almost surely. We first explain why $\Lambda \leqslant \lambda_c$. The peeling equation for $p = v = 1$ shows that

$$
\sum_{j=0}^{+\infty} |\mathcal{T}_1(j)| a^1_{j+1} \leqslant a^1_1 < +\infty.
$$

On the other hand, we know that $|\mathcal{T}_1(j)| \sim c\lambda_c^{-j} j^{-5/2}$ as $j \to +\infty$ for some constant $c > 0$. Therefore, if $\mathbb{P}\left(\Lambda \geqslant \lambda_c + \varepsilon\right) \geqslant \varepsilon$ for some $\varepsilon > 0$, then $a^1_{v+1} \geqslant \varepsilon(\lambda_c + \varepsilon)^v$ for every v and the series above diverge, so we get a contradiction.

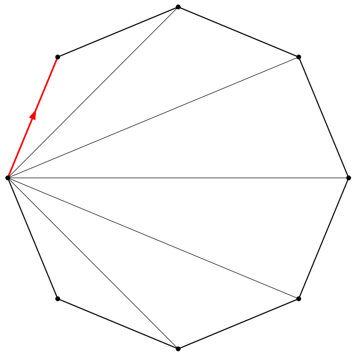


Figure 6.5 – The "triangle chain" triangulation t_p with p vertices (for $p = 8$).

We finally prove that $\mathbb{P}(\Lambda = 0) = 0$. As explained above, we already know that

$$a_v^p = \int_0^{\lambda_c} a_v^p(\lambda)\mu(\mathrm{d}\lambda),$$

where μ is the distribution of Λ. We also have $a_v^p(0) = \mathbb{1}_{p=v}$. Therefore, if we set $\delta = \mathbb{P}(\Lambda = 0)$, for every $p \geqslant 1$, we have $a_p^p \geqslant \delta$. Now let t_p be the triangulation with p vertices and a hole of perimeter p represented on Figure 6.5. For every $p \geqslant 1$, we have $\mathbb{P}(t_p \subset T) = a_p^p \geqslant \delta$. Since the events $t_p \subset T$ are nonincreasing in p, we have

$$\mathbb{P}(\forall p \geqslant 1, t_p \subset T) \geqslant \delta.$$

On the other hand, if $t_p \subset T$, then the degree of the root vertex is at least $p + 2$, so we have $\mathbb{P}(\deg(\rho) = +\infty) \geqslant \delta$. Since the degrees are finite, we must have $\delta = 0$, so Λ is supported in $(0, \lambda_c]$, which concludes the proof. $\square$

6.4 Ergodicity

In all this section, we denote by (g_n) a sequence such that $\frac{g_n}{n} \to \theta \in \left[0, \frac{1}{2}\right)$ and by T a subsequential limit of (T_{n,g_n}) for d_{loc}. In order to keep the notation light, we will always implicitly restrict ourselves to values of n along which T_{n,g_n} converges to T in distribution.

For any n, the triangulation T_{n,g_n} is weakly Markovian, so this is also the case of T. Therefore, by Theorem 6.0.2, we know that T must be a mixture of PSHT, i.e. there is a random variable $\Lambda \in (0, \lambda_c]$ such that T has the same distribution as the PSHT with random parameter $\mathbb{T}_\Lambda$.

We also note right now that by the discussion in the end of Section 6.1.3 (Equation (6.1.5)), the parameter Λ is a measurable function of the triangulation $\mathbb{T}_\Lambda$. More precisely, if P^Λ and V^Λ are the perimeter and volume processes associated to the peeling exploration of $\mathbb{T}_\Lambda$, then Λ can be defined as $f^{-1}\left(\lim_{i\to+\infty}\frac{P^\Lambda(i)}{V^\Lambda(i)}\right)$, where $f(\lambda)=1-4h$ is injective. In particular Λ is defined without any ambiguity. Our goal is now to prove that Λ is deterministic, and given by (6.0.2).

6.4.1 The two holes argument

Roughly speaking, we know that T_{n,g_n} seen from a typical point e_n looks like a PSHT with random parameter Λ. We would like to show that Λ does not depend on (T_{n,g_n}, e_n). The first step is essentially to prove that Λ does not depend on e_n.

More precisely, conditionally on T_{n,g_n}, we pick two independent uniform oriented edges e_n^1, e_n^2 of T_{n,g_n}. The pairs (T_{n,g_n}, e_n^1) and (T_{n,g_n}, e_n^2) have the same distribution, and both converge in distribution to $\mathbb{T}_\Lambda$. It follows that the pairs

$$\left((T_{n,g_n}, e_n^1), (T_{n,g_n}, e_n^2)\right)$$

for $n \geqslant 1$ are tight, so up to further extraction, they converge to a pair $(\mathbb{T}^1_{\Lambda_1}, \mathbb{T}^2_{\Lambda_2})$, where both marginals have the same distribution as $\mathbb{T}_\Lambda$. By the above discussion, the variables Λ_1 and Λ_2 are well-defined. By the Skorokhod representation theorem, we may assume the convergence in distribution is almost sure. Our goal in this subsection is to prove the following lemma.

Proposition 6.4.1. *We have $\Lambda_1 = \Lambda_2$ almost surely.*

Proof. The idea of the proof is the following: if $\Lambda_1 \neq \Lambda_2$, consider two large neighbourhoods N_n^1 and N_n^2 (say, of size 100) with the same perimeter around e_n^1 and e_n^2. Then there is a natural way to "swap" N_n^1 and N_n^2 in T_{n,g_n} (cf. Figure 6.6), without changing its distribution. If we do this, then T_{n,g_n} around e_n^1 looks like $\mathbb{T}_{\Lambda_1}$ in a neighbourhood of the root of size 100, and like $\mathbb{T}_{\Lambda_2}$ out of this neighbourhood. However, if $\Lambda_1 \neq \Lambda_2$, such a configuration is highly unlikely for a mixture of PSHT.

More precisely, in all the proof that follows, we consider a deterministic peeling algorithm $\mathcal{A}$, according to which we will explore T_{n,g_n} around e_1^n and around e_2^n. All the explorations we will consider will be filled-in: every time the peeled face separates the undiscovered part in two, we discover completely the smaller part. While the computations in the beginning of the proof hold for any choice of $\mathcal{A}$, we will need later to specify the choice of the algorithm.

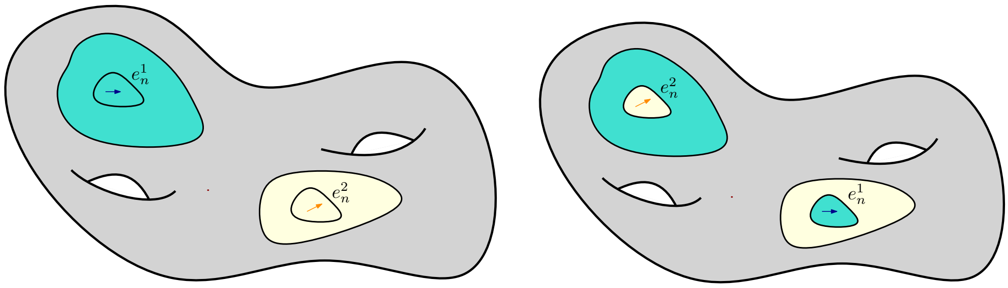


Figure 6.6 – The idea of the two-holes argument: the inner parts around e_n^1 and e_n^2 are "swapped". The root edges e_n^1 and e_n^2 are swapped at the same time as their neighbourhoods.

We denote by $\mathcal{E}_n^1(i)$ the explored part at time i when the exploration is started from e_1^n. If at some point $\mathcal{E}_n^1(i)$ is non-planar, then the exploration is stopped. We write respectively $P_n^1(i)$ and $V_n^1(i)$ for its perimeter and its volume (i.e. its total number of vertices). For any $p \geqslant 2$, let $\tau_n^1(p)$ be the smallest i such that $P_n^1(i) = p$. We denote by $\mathcal{E}_n^2(i)$, $P_n^2(i)$, $V_n^2(i)$ and $\tau_n^2(p)$ the analog quantities for the exploration started from e_n^2. We also denote by $\mathcal{E}_\infty^1(i)$, $P_\infty^1(i)$, $V_\infty^1(i)$ and $\tau_\infty^1(p)$ the analog quantities for $\mathbb{T}_{\Lambda_1}^1$, and similarly for $\mathbb{T}_{\Lambda_2}^2$.

We fix $\varepsilon > 0$. We recall that

$$\frac{P_\infty^1(i)}{V_\infty^1(i)} \xrightarrow[i\to+\infty]{a.s.} f(\Lambda_1),$$

where $f : (0, \lambda_c] \to [0, 1)$ is bijective and decreasing. Note also that the times $\tau_\infty^1(p)$ are a.s. finite and $\tau_\infty^1(p) \to +\infty$ when $p \to +\infty$, so for p large enough, we have

$$\mathbb{P}\left(\left|\frac{p}{V_\infty^1(\tau_\infty^1(p))} - f(\Lambda_1)\right| \geqslant \varepsilon\right) \leqslant \varepsilon.$$

We fix such a p until the end of the proof. Moreover, we have

$$\frac{q}{V_\infty^1(\tau_\infty^1(q)) - V_\infty^1(\tau_\infty^1(p))} \xrightarrow[q\to+\infty]{a.s.} f(\Lambda_1),$$

so for q large enough, we have

$$\mathbb{P}\left(\left|\frac{q}{V_\infty^1(\tau_\infty^1(q)) - V_\infty^1(\tau_\infty^1(p))} - f(\Lambda_1)\right| \geqslant \varepsilon\right) \leqslant \varepsilon.$$

We fix a such a $q > p$ until the end of the proof. Moreover, the local convergence of (T_{n,g_n}, e_n^1) to $\mathbb{T}_{\Lambda_1}^1$ implies the convergence of the peeling exploration.

Hence, the probability that $\tau_n^1(p)$ is finite goes to 1 as $n \to +\infty$. Therefore, for n large enough, we have

$$\mathbb{P}\left(\left|\frac{p}{V_n^1(\tau_n^1(p))} - f(\Lambda_1)\right| \geqslant 2\varepsilon\right) \leqslant 2\varepsilon, \tag{6.4.1}$$

$$\mathbb{P}\left(\left|\frac{q}{V_n^1(\tau_n^1(q)) - V_n^1(\tau_n^1(p))} - f(\Lambda_1)\right| \geqslant 2\varepsilon\right) \leqslant 2\varepsilon. \tag{6.4.2}$$

By combining (6.4.1) and (6.4.2), we also obtain

$$\mathbb{P}\left(\left|\frac{q}{V_n^1(\tau_n^1(q)) - V_n^1(\tau_n^1(p))} - \frac{p}{V_n^1(\tau_n^1(p))}\right| \geqslant 4\varepsilon\right) \leqslant 4\epsilon \tag{6.4.3}$$

for n large enough. Moreover, the same is true if we replace the exploration from e_n^1 by the exploration from e_n^2 (with of course Λ_2 playing the role of Λ_1).

We now specify the properties that we need our peeling algorithm $\mathcal{A}$ to satisfy. It roughly means that the edges that we peel after time $\tau(p)$ should not depend on what happened before time $\tau(p)$. More precisely, before the time $\tau(p)$, the peeled edge can be any edge of the boundary. At time $\tau(p)$, we color in red an edge of the boundary of $\mathcal{E}(\tau(p))$ according to some deterministic convention. At time $i \geqslant \tau(p)$, the choice of the edge to peel at time $i+1$ must only depend on the triangulation $\mathcal{E}(i) \backslash \mathcal{E}(\tau(p))$, which is rooted at the red edge (see Figure 6.7). It is easy to see that such an algorithm exists: we only need to fix a peeling algorithm $\mathcal{A}'$ for triangulations of the sphere or the plane, and a peeling algorithm $\mathcal{A}''$ for triangulations of the p-gon, and to use $\mathcal{A}'$ before time $\tau(p)$ and $\mathcal{A}''$ after $\tau(p)$. Note also that we can know, by only looking at $\mathcal{E}(i)$, if $i \leqslant \tau(p)$ or not, so at each step the peeled edge will indeed depend only on the explored part.

We can now define precisely our surgery operation on $(T_{n,g_n}, e_n^1, e_n^2)$. We say that e_n^1 and e_n^2 are *well separated* if $\tau_n^1(p), \tau_n^2(p) < +\infty$ and if the regions $\mathcal{E}_n^1(\tau_n^1(p))$ and $\mathcal{E}_n^2(\tau_n^2(p))$ have no common vertex. If e_n^1 and e_n^2 are well separated, we remove the triangulations $\mathcal{E}_n^1(\tau_n^1(p))$ and $\mathcal{E}_n^2(\tau_n^2(p))$, which creates two holes of perimeter p. We then glue each of the two triangulations to the hole where the other was, in such a way that the red edges of our peeling algorithm coincide (see Figure 6.6). If the two roots are not well-separated, then $(T_{n,g_n}, e_n^1, e_n^2)$ is left unchanged. This operation is an involution on the set of bi-rooted triangulations with fixed size and genus. Therefore, if we denote by T'_{n,g_n} the new triangulation we obtain, it is still uniform, and $\left(T'_{n,g_n}, e_n^1, e_n^2\right)$ has the same distribution as $(T_{n,g_n}, e_n^1, e_n^2)$.

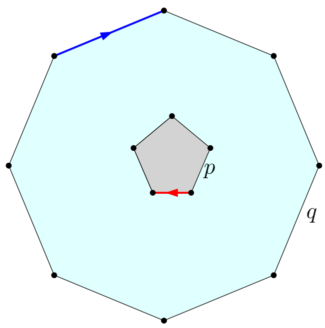


Figure 6.7 – Illustration of the choice of the peeling algorithm: in gray, the part $\mathcal{E}(\tau(p))$. In cyan, the part $\mathcal{E}(i)\backslash\mathcal{E}(\tau(p))$, rooted at the red edge. The peeled edge (in blue) may only depend on the cyan part (which is rooted at the red edge).

Lemma 6.4.2. *The probability that e_n^1 and e_n^2 are well separated goes to* 1 *as $n \to +\infty$.*

Proof. The local convergence of the exploration implies that the sequence $(\tau_n^1(p))_{n\geqslant 0}$ is tight, and so is $(V_n^1(\tau_n^1(p)))_{n\geqslant 0}$. Since the diameter of a triangulation is bounded by its volume, the diameters of the maps $\mathcal{E}_n^1(\tau_n^1(p))$ are tight (when $n \to +\infty$), and the same is true around e_2^n. On the other hand, by local convergence (for d_{loc}), for every fixed r, the volume of the ball of radius r around e_n^1 is tight. Since e_n^2 is uniform and independent of e_n^1, we have

$$\mathbb{P}\left(d_{T_{n,g_n}}(e_n^1, e_n^2) \leqslant r\right) \xrightarrow[n\to+\infty]{} 0$$

for every $r \geqslant 0$. From here, we obtain

$$\mathbb{P}\left(d_{T_{n,g_n}}(e_n^1, e_n^2) \leqslant \mathrm{diam}\left(\mathcal{E}_n^1(\tau_n^1(p))\right) + \mathrm{diam}\left(\mathcal{E}_n^2(\tau_n^2(p))\right)\right) \xrightarrow[n\to+\infty]{} 0.$$

If this last event does not occur, then the regions $\mathcal{E}_n^1(\tau_n^1(p))$ and $\mathcal{E}_n^2(\tau_n^2(p))$ do not intersect, which proves the lemma. $\square$

Now, if we perform a peeling exploration on T'_{n,g_n} from e_n^1 using algorithm $\mathcal{A}$, then the part explored at time $\tau(p)$ will be exactly the triangulation $\mathcal{E}_n^1(\tau_n^1(p))$. Moreover, the red edge on $\partial\mathcal{E}_n^1(\tau_n^1(p))$ is glued at the same place as the red edge on $\partial\mathcal{E}_n^2(\tau_n^2(p))$ and our peeling algorithm "forgets" the interior of $\mathcal{E}_n^1(\tau_n^1(p))$. Hence, the part explored between $\tau(p)$ and $\tau(q)$ is exactly

$\mathcal{E}_n^2(\tau_n^2(q))\backslash\mathcal{E}_n^2(\tau_n^2(p))$. Therefore, the part discovered between times $\tau(p)$ and $\tau(q)$ has perimeter q and volume

$$V_n^2(\tau_n^2(q)) - V_n^2(\tau_n^2(p)).$$

Now, since (T'_{n,g_n}, e_n^1) has the same distribution as (T_{n,g_n}, e_n^1), we can apply (6.4.3) to the exploration of T'_{n,g_n} from e_n^1. We obtain

$$\mathbb{P}\left(\left|\frac{q}{V_n^2(\tau_n^2(q)) - V_n^2(\tau_n^2(p))} - \frac{p}{V_n^1(\tau_n^1(p))}\right| \geqslant 4\varepsilon\right) \leqslant 4\epsilon.$$

By combining this with (6.4.1) for the exploration of T_{n,g_n} started from e_n^1 and with (6.4.2) for the exploration of T_{n,g_n} started from e_n^2, we obtain

$$\mathbb{P}\left(|f(\Lambda_1) - f(\Lambda_2)| \geqslant 8\varepsilon\right) \leqslant 8\varepsilon.$$

This is true for any $\varepsilon > 0$, so we have $f(\Lambda_1) = f(\Lambda_2)$ a.s.. Since f is strictly decreasing on $(0, \Lambda_c]$, we are done. □

6.4.2 Conclusion

In order to conclude the proof of Theorem 6.0.1, we will need to compute the mean inverse root degree in the PSHT. This is the only observable that is easy to compute in T_{n,g_n}, so it is not surprising that such a result is needed to link θ to λ.

Proposition 6.4.3. *For $\lambda \in (0, \lambda_c]$, let*

$$d(\lambda) = \mathbb{E}\left[\frac{1}{\deg_{\mathbb{T}_\lambda}(\rho)}\right].$$

Then we have

$$d(\lambda) = \frac{h \log \frac{1+\sqrt{1-4h}}{1-\sqrt{1-4h}}}{(1+8h)\sqrt{1-4h}},$$

where h is given by (6.1.3). In particular, the function d is strictly increasing on $(0, \lambda_c]$, with $d(\lambda_c) = \frac{1}{6}$ and $\lim_{\lambda\to 0} d(\lambda) = 0$.

We postpone the proof of Proposition 6.4.3 until Section 6.4.3, and finish the proof of Theorem 6.0.1. We recall that we work with a subsequential limit of (T_{n,g_n}), and we know that it has the same distribution as $\mathbb{T}_\Lambda$ for some random Λ. Our goal is to prove that Λ is the solution of $d(\Lambda) = \frac{1-2\theta}{6}$.

Note that by Proposition 6.4.3, the solution of this equation exists and is unique.

The idea is the following: by Proposition 6.4.1, the random parameter Λ only depends on the triangulation T_{n,g_n} and not on its root. On the one hand, for any triangulation, the mean inverse root degree over all possible choices of the root is $\frac{1-2\theta}{6}$. Therefore, the mean inverse root degree over all triangulations corresponding to a fixed value of Λ is $\frac{1-2\theta}{6}$. On the other hand, the mean inverse degree conditionally on Λ is $d(\Lambda)$, so we should have $d(\Lambda)=\frac{1-2\theta}{6}$.

To prove this properly, we need a way to "read" Λ on finite maps, which is the goal of the next (easy) technical lemma. As earlier, we restrict ourselves to a subset of values of n along which $T_{n,g_n} \to \mathbb{T}_\Lambda$.

Lemma 6.4.4. *There is a function $\ell : \bigcup_{n \geqslant 1} \mathcal{T}(n,g_n) \longrightarrow [0,\lambda_c]$ such that we have the convergence*

$$(T_{n,g_n}, \ell(T_{n,g_n})) \xrightarrow[n\to+\infty]{(d)} (\mathbb{T}_\Lambda, \Lambda)\,.$$

Note that a priori $\ell(t)$ may depend on the choice of the root of t, although Proposition 6.4.1 will guarantee that this is asymptotically not the case.

Proof. By the Skorokhod representation theorem, we may assume the convergence $T_{n,g_n} \to \mathbb{T}_\Lambda$ is almost sure. As explained in the beginning of Section 6.4, the parameter Λ is a measurable function of $\mathbb{T}_\Lambda$, so there is a measurable function $\widetilde{\ell}$ from the set of all (finite or infinite) triangulations such that $\widetilde{\ell}(\mathbb{T}_\Lambda)=\Lambda$ a.s.. Therefore, for any $\varepsilon>0$, we can find a continuous function (for the local topology) ℓ_ε such that

$$\mathbb{P}\left(|\ell_\varepsilon(\mathbb{T}_\Lambda)-\Lambda| \geqslant \frac{\varepsilon}{2}\right) \leqslant \frac{\varepsilon}{2}.$$

For n larger than some N_ε, we have

$$\mathbb{P}\left(|\ell_\varepsilon(T_{n,g_n})-\Lambda| \geqslant \varepsilon\right) \leqslant \varepsilon.$$

Now let $\varepsilon_k \to 0$ as $k \to +\infty$. We may assume that (N_{ε_k}) is strictly increasing. For any triangulation $t \in \mathcal{T}(n,g_n)$, we set $\ell(t)=\ell_{\varepsilon_k}(t)$, where k is such that $N_{\varepsilon_k} < n \leqslant N_{\varepsilon_{k+1}}$. This ensures $\ell(T_{n,g_n}) \to \Lambda$ almost surely, so

$$(T_{n,g_n}, \ell(T_{n,g_n})) \xrightarrow[n\to+\infty]{(d)} (\mathbb{T}_\Lambda, \Lambda)\,.$$

□

We are now ready to prove our main Theorem.

Proof of Theorem 6.0.1. Let f be a continuous, bounded function on $[0,\lambda_c]$. For any n, let e_n^1 and e_n^2 be two oriented edges chosen independently and uniformly in T_{n,g_n}, and let ρ_n^1 and ρ_n^2 be their starting points. We will estimate in two different ways the limit of the quantity

$$\mathbb{E}\left[\frac{1}{\deg(\rho_n^1)}f(\ell(T_{n,g_n},e_n^1))\right]. \tag{6.4.4}$$

On the one hand, the quantity in the expectation is bounded and converges in distribution to $\frac{1}{\deg(\rho)}f(\Lambda)$, where ρ is the root vertex of $\mathbb{T}_\Lambda$, so (6.4.4) goes to

$$\mathbb{E}\left[\frac{1}{\deg(\rho)}f(\Lambda)\right]=\mathbb{E}\left[d(\Lambda)f(\Lambda)\right]$$

as $n\to+\infty$. On the other hand, by Proposition 6.4.1 and Lemma 6.4.4, we have

$$\left(\ell(T_{n,g_n},e_n^1),\ell(T_{n,g_n},e_n^2)\right)\xrightarrow[n\to+\infty]{(d)}(\Lambda,\Lambda),$$

so $\ell(T_{n,g_n},e_n^1)-\ell(T_{n,g_n},e_n^2)$ goes to 0 in probability. Since f is bounded and uniformly continuous, the difference $f\left(\ell(T_{n,g_n},e_n^1)\right)-f\left(\ell(T_{n,g_n},e_n^2)\right)$ goes to 0 in L^1, so we have

$$\mathbb{E}\left[\frac{1}{\deg(\rho_n^1)}f(\ell(T_{n,g_n},e_n^1))\right]=\mathbb{E}\left[\frac{1}{\deg(\rho_n^1)}f(\ell(T_{n,g_n},e_n^2))\right]+o(1).$$

Moreover, the expectation of $\frac{1}{\deg(\rho_n^1)}$ conditionally on (T_{n,g_n},e_n^2) is equal to $\frac{|V(T_{n,g_n})|}{2|E(T_{n,g_n})|}=\frac{n+2-2g_n}{6n}$, so we can write

$$\begin{aligned}\mathbb{E}\left[\frac{1}{\deg(\rho_n^1)}f(\ell(T_{n,g_n},e_n^1))\right]&=\frac{n+2-2g_n}{6n}\mathbb{E}\left[f(\ell(T_{n,g_n},e_n^2))\right]+o(1)\\&\xrightarrow[n\to+\infty]{}\frac{1-2\theta}{6}\mathbb{E}\left[f(\Lambda)\right].\end{aligned}$$

Therefore, for any bounded, continuous f on $[0,\lambda_c]$, we have

$$\mathbb{E}\left[d(\Lambda)f(\Lambda)\right]=\frac{1-2\theta}{6}\mathbb{E}\left[f(\Lambda)\right],$$

so $d(\Lambda)=\frac{1-2\theta}{6}$ a.s.. By injectivity of d, this fixes a deterministic value for Λ that depends only on θ, which concludes the proof. $\square$

6.4.3 The average root degree in the type-I PSHT via uniform spanning forests

Our goal is now to prove Proposition 6.4.3. Since we have not been able to perform a direct computation, our strategy will be to use the similar computation for the type-II PSHIT that is done in [Bud18a, Appendix B]. To link the mean degree in the type-I and in the type-II PSHT, we use the core decomposition of the type-I PSHT [Bud18a, Appendix A] and an interpretation in terms of the Wired Uniform Spanning Forest.

The type-II case. We will need to use the computation of the same quantity in the type-II case, which is performed in [Bud18a, Appendix B]. We first recall briefly the definition of the type-II PSHT [Cur16]. These are the type-II analogs of the type-I PSHT, i.e. they may contain multiple edges but no loop. They form a one-parameter family $\left(\mathbb{T}_\kappa^{II}\right)_{0<\kappa\leqslant\kappa_c}$ of random infinite triangulations of the plane, where $\kappa_c = \frac{2}{27}$. Their distribution is characterized as follows: for any type-II triangulation with a hole of perimeter p and v vertices in total, we have

$$\mathbb{P}\left(t \subset \mathbb{T}_\kappa^{II}\right) = C_p^{II}(\kappa) \times \kappa^v,$$

where the numbers $C_p^{II}(\kappa)$ are explicitly determined by κ.

Proposition 6.4.5. *[Bud18a, Appendix B] For $\kappa \in (0, \kappa_c]$, let*

$$d_{II}(\kappa) = \mathbb{E}\left[\frac{1}{\deg_{\mathbb{T}_\kappa^{II}}(\rho)}\right].$$

Then we have

$$d_{II}(\kappa) = -\frac{1-\alpha}{2} + \frac{(1-\alpha)\sqrt{\alpha}}{2\sqrt{3\alpha-2}} \log \frac{\sqrt{\alpha}+\sqrt{3\alpha-2}}{\sqrt{\alpha}-\sqrt{3\alpha-2}},$$

where $\alpha \in \left[\frac{2}{3}, 1\right)$ is such that $\kappa = \frac{\alpha^2(1-\alpha)}{2}$.

In [Bud18a], Proposition 6.4.5 is proved by using a peeling exploration to obtain an equation on the generating function of the mean inverse root degree in PSHT with boundaries. While this approach could theoretically also work in the type-I setting, some technical details make the formulas much more complicated. Although everything remains in theory completely solvable, our efforts to push the computation until the end have failed, even with a computer algebra software. Therefore, we will rely on Proposition 6.4.5 and on the correspondence between type-I PSHT and type-II PSHT given in [Bud18a, Appendix A].

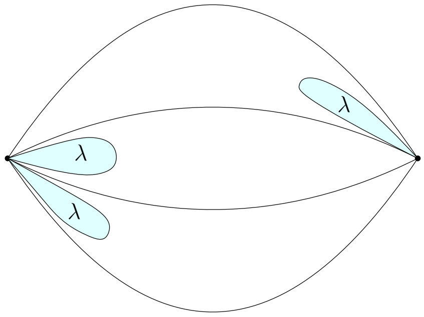


Figure 6.8 – The random block A_λ (here we have $B = 4$). The green parts are i.i.d. λ-Boltzmann triangulations of the 1-gon.

The type-I–type-II correspondence. We now introduce the correspondence between the type-I and the type-II PSHT stated in [Bud18a, Appendix A]. We write $\beta = 1 - \frac{1+2h}{\sqrt{1+8h}}$, and $\kappa = \frac{h}{(1+2h)^3}$, where h is given by (6.1.3). We also define a random triangulation A_λ of the 2-gon as follows. We start from two vertices x and y linked by B edges, where $B-1$ is a geometric variable of parameter β. In each of the $B-1$ slots between these edges, we insert a loop (the loops are glued to x or y according to independent fair coin flips). Finally, these $B-1$ loops are filled with i.i.d. Boltzmann triangulations of the 1-gon with parameter λ (see Figure 6.8). Copies of A_λ will be called *blocks* in what follows.

Let $|A_\lambda|$ be the number of inner vertices of A_λ (i.e. x and y excluded), and let $E(A_\lambda)$ be its set of edges (including the boundary edges). The Euler formula implies $|E(A_\lambda)| = 1+3|A_\lambda|$. Let $\widetilde{A}_\lambda$ be a random triangulation with the same distribution as A_λ, biased by $|E(A_\lambda)|$. After elementary computations, the correspondence shown in [Bud18a, Appendix A] can be reformulated as follows.

Proposition 6.4.6. *The triangulation $\mathbb{T}_\lambda$ has the same distribution as $\mathbb{T}_\kappa^{II}$, where:*

- *the root edge has been replaced by an independent copy of $\widetilde{A}_\lambda$, rooted at a uniform oriented edge of $\widetilde{A}_\lambda$;*
- *all the other edges of $\mathbb{T}_\kappa^{II}$ have been replaced by i.i.d. copies of A_λ.*

In all the rest of this section, we will consider that $\mathbb{T}_\lambda$ and $\mathbb{T}_\kappa^{II}$ are coupled together as described in Proposition 6.4.6. For each edge e of $\mathbb{T}_\kappa^{II}$, we will denote by A_λ^e the block replacing e. It will also be useful to consider the triangulation obtained from $\mathbb{T}_\kappa^{II}$ by replacing all the edges (including the root) by i.i.d. copies of A_λ. We denote this triangulation by $\mathbb{T}'_\lambda$. We also note right now that $\mathbb{T}_\lambda$ is absolutely continuous with respect to $\mathbb{T}'_\lambda$.

The wired uniform spanning forest. The last ingredient that we need to prove Proposition 6.4.3 is the oriented wired uniform spanning forest (OWUSF) [BLPS01], which may be defined on any transient graph. We first recall its definition. We recall that if (γ_i) is a transient path, its *loop-erasure* is the path (γ_{i_j}), where $i_0 = 0$ and $i_{j+1} = 1 + \max\{i \geqslant i_j | \gamma(i) = \gamma(i_j)\}$. If G is an infinite transient graph, its OWUSF $\overrightarrow{F}$ is generated by the following generalization of the Wilson algorithm [Wil96].

1. We list the vertices of G as $(v_n)_{n \geqslant 1}$.
2. We run a simple random walk (X_n) on G started from v_1, and consider its loop-erasure, which is an oriented path from v_1 to ∞. We add this path to $\overrightarrow{F}$.
3. We consider the first i such that v_i does not belong to any edge of $\overrightarrow{F}$ yet. We run a SRW from v_i, stopped when it hits $\overrightarrow{F}$, and add its loop-erasure to $\overrightarrow{F}$.
4. We repeat the step 3 infinitely many times.

It is immediate that $\overrightarrow{F}$ is an oriented forest such that for any vertex v of G, there is exactly one oriented edge of $\overrightarrow{F}$ going out of v. It can be checked that the distribution of $\overrightarrow{F}$ does not depend on the way the vertices are ordered.

Stationarity and reversibility. We recall that a random rooted graph G is *reversible* if its distribution is invariant under reversing the orientation of the root edge. It is *stationary* if its distribution is invariant under re-rooting G at the first step of the simple random walk. In particular, we recall from [Cur16] that the type-II PSHT are stationary and reversible. Moreover, the proof of [Cur16] also works for the type-I PSHT.

If two random, rooted transient graphs $(G, \overrightarrow{e_0})$ and $(G', \overrightarrow{e_0}')$ have the same distribution and if $\overrightarrow{F}$ and $\overrightarrow{F}'$ are their respective OWUSF, then $(G, \overrightarrow{e_0}, \overrightarrow{F})$ and $(G', \overrightarrow{e_0}', \overrightarrow{F}')$ also have the same distribution. It follows that if $(G, \overrightarrow{e_0})$ is stationary and reversible, so is the triplet $(G, \overrightarrow{e_0}, \overrightarrow{F})$. The link between the OWUSF and the mean inverse root degree is given by the next lemma.

Lemma 6.4.7. *Let $(G, \overrightarrow{e_0})$ be a stationary, reversible and transient random rooted graph, and let ρ be the starting point of the root edge $\overrightarrow{e_0}$. We denote by $\overrightarrow{F}$ the OWUSF of G. Then*

$$\mathbb{P}\left(\overrightarrow{e_0} \in \overrightarrow{F}\right) = \mathbb{E}\left[\frac{1}{\deg_G(\rho)}\right].$$

Proof. Conditionally on G, let $\overrightarrow{e_1}$ be a random oriented edge chosen uniformly (independently of $\overrightarrow{F}$) among all the edges leaving the root vertex. By invariance and reversibility of $(G, \overrightarrow{e_0})$, we know that $(G, \overrightarrow{e_1}, \overrightarrow{F})$ has the same distribution as $(G, \overrightarrow{e_0}, \overrightarrow{F})$. Therefore, we have

$$\mathbb{P}\left(\overrightarrow{e_0} \in \overrightarrow{F}\right) = \mathbb{P}\left(\overrightarrow{e_1} \in \overrightarrow{F}\right) = \mathbb{E}\left[\mathbb{P}\left(\overrightarrow{e_1} \in \overrightarrow{F}|G, \overrightarrow{e_0}, \overrightarrow{F}\right)\right] = \mathbb{E}\left[\frac{1}{\deg_G(\rho)}\right],$$

where in the end we use the fact that, by construction, $\overrightarrow{F}$ contains exactly one oriented edge going out of the root vertex. □

Remark 6.4.8. *Lemma 6.4.7 is the analog for stationary, reversible graphs of the fact that the WUSF on any unimodular random graph has expected root degree 2 [AL07, Theorem 7.3]. These two properties can also be deduced from one another.*

Proof of Proposition 6.4.3. By an easy argument of uniform integrability, the function d is continuous in λ, so it is enough to prove the formula for $\lambda < \lambda_c$. In this case, all variants of $\mathbb{T}_\lambda$ that we will consider are a.s. transient by results from [Cur16], so the OWUSF on these graphs is well-defined. We now fix $\lambda \in (0, \lambda_c)$. Let β and κ be given by Proposition 6.4.6. We will denote by $\overrightarrow{e_0}$, $\overrightarrow{e_0}'$ and $\overrightarrow{e_0}^{II}$ the respective root edges of $\mathbb{T}_\lambda$, $\mathbb{T}'_\lambda$ and $\mathbb{T}^{II}_\kappa$, and by $\overrightarrow{F}$, $\overrightarrow{F}'$ and $\overrightarrow{F}^{II}$ their respective OWUSF.

By Proposition 6.4.6, conditionally on $\mathbb{T}^{II}_\kappa$ and on the blocks $(A^e_\lambda)_{e\in E(\mathbb{T}^{II}_\kappa)}$, the edge $\overrightarrow{e_0}$ is picked uniformly among the oriented edges of $A^{e_0^{II}}_\lambda$. Moreover, this choice is also independent of $\overrightarrow{F}$. Therefore, we have

$$\mathbb{P}\left(\overrightarrow{e_0} \in \overrightarrow{F}\right) = \mathbb{E}\left[\mathbb{P}\left(\overrightarrow{e_0} \in \overrightarrow{F}|\mathbb{T}^{II}_\kappa, (A^e_\lambda), \overrightarrow{F}\right)\right] = \mathbb{E}\left[\frac{|E(\widetilde{A}_\lambda) \cap \overrightarrow{F}|}{2|E(\widetilde{A}_\lambda)|}\right],$$

where we recall that $E(\widetilde{A}_\lambda)$ is the set of edges of $\widetilde{A}_\lambda$. Moreover, the pair $(\mathbb{T}_\lambda, \overrightarrow{F})$ has the same distribution as $(\mathbb{T}'_\lambda, \overrightarrow{F}')$ biased by $|E(A^{e_0}_\lambda)|$. Therefore, we can write

$$\mathbb{P}\left(\overrightarrow{e_0} \in \overrightarrow{F}\right) = \frac{\mathbb{E}\left[|E(A^{e_0}_\lambda) \cap \overrightarrow{F}'|\right]}{2\mathbb{E}\left[|E(A_\lambda)|\right]}. \tag{6.4.5}$$

Since the denominator can easily be explicitly computed by using the definition of the block A_λ, we first focus on the numerator. In a block, we call *principal edges* the edges joining the two boundary vertices of the block. We know that there is exactly one edge of $\overrightarrow{F}'$ going out of every inner vertex of A_λ. Moreover, if a non-principal edge of A_λ belongs to $\overrightarrow{F}'$, then it goes out of an inner vertex (it cannot go out of a boundary vertex since the edges of $\overrightarrow{F}'$ are "directed towards infinity"). Therefore, the number of non-principal edges of A_λ that belong to $\overrightarrow{F}'$ is exactly $|A_\lambda|$. On the other hand, since $\overrightarrow{F}'$ is a forest, it contains at most one principal edge of A_λ. Therefore, we have

$$\mathbb{E}\left[|E(A_\lambda^{e_0}) \cap \overrightarrow{F}'|\right] = \mathbb{E}[|A_\lambda|] + \mathbb{P}\left(\overrightarrow{F}' \text{ contains a principal edge of } A_\lambda^{e_0}\right).$$

To finish the computation, we claim that

$$\mathbb{P}\left(\overrightarrow{F}' \text{ contains a principal edge of } A_\lambda^{e_0}\right) = 2\mathbb{E}\left[\frac{1}{\deg_{\mathbb{T}_\kappa^{II}}(\rho)}\right] = 2d_{II}(\kappa). \tag{6.4.6}$$

Indeed, let ρ^{II} be the starting vertex of $\overrightarrow{e_0}^{II}$. By reversibility of $\mathbb{T}_\kappa^{II}$, the left-hand side of (6.4.6) is twice the probability that $\overrightarrow{F}'$ contains a principal edge of $A_\lambda^{e_0}$ going out of ρ^{II}. Moreover, since $\mathbb{T}_\kappa^{II}$ is stationary, the distribution of $\mathbb{T}'_\lambda$ is stable under the following operation:

- first, we resample the root edge of $\mathbb{T}_\kappa^{II}$ uniformly among all the oriented edges going out of ρ^{II};
- then, we pick the new root edge of $\mathbb{T}'_\lambda$ uniformly among all the oriented edges of the block corresponding to the new root edge of $\mathbb{T}_\kappa^{II}$.

Note that this operation is different from resampling the root of $\mathbb{T}'_\lambda$ uniformly among the edges going out of its root vertex, and that $\mathbb{T}'_\lambda$ is *not* stationary.

Since $\mathbb{T}'_\lambda$ is stationary for this operation, so is $\left(\mathbb{T}'_\lambda, \overrightarrow{F}'\right)$. On the other hand, there is exactly one block incident to ρ^{II} in which a principal edge going out of ρ^{II} belongs to $\overrightarrow{F}'$. By the same argument as in the proof of Lemma 6.4.7, this implies (6.4.6). By combining Lemma 6.4.7 with (6.4.5) and (6.4.6), we obtain

$$d(\lambda) = \frac{\mathbb{E}[|A_\lambda|] + 2d_{II}(\kappa)}{2\left(1 + 3\mathbb{E}[|A_\lambda|]\right)}.$$

By the definition of A_λ, we also have

$$\mathbb{E}[|A_\lambda|] = \frac{\beta}{1-\beta} \times \frac{\lambda Z_1'(\lambda)}{Z_1(\lambda)} = \frac{2h}{1+2h},$$

where $Z_1(\lambda)$ is the partition function of λ-Boltzmann type-I triangulations of a 1-gon. Finally, the value of $d_{II}(\kappa)$ is given by Proposition 6.4.5. It is easy to obtain that the α of Proposition 6.4.5 is equal to $\frac{1}{1+2h}$, and we get the formula for $d(\lambda)$.

It remains to prove that this is an increasing function of λ (or equivalently of h). For this, set $x = \sqrt{1-4h}$. We want to prove that the function

$$x \to \frac{1-x^2}{4x(3-2x^2)} \log \frac{1+x}{1-x}$$

is decreasing on $[0,1]$. By computing the derivative with respect to x, this is equivalent to

$$(2x^4 - 3x^2 + 3)x \log \frac{1+x}{1-x} \geqslant 2x^2(3-2x^2)$$

for $0 < x < 1$. The power series expansion of the left-hand side is

$$6x^2 - 4x^4 + \sum_{k \geqslant 3} \left(\frac{6}{2k-1} - \frac{6}{2k-3} + \frac{4}{2k-5} \right) x^{2k} \geqslant 6x^2 - 4x^4.$$

Since all the terms in the sum are positive, this proves monotonicity. □

6.5 Asymptotic enumeration

The goal of this section is to prove Theorem 6.0.3. The proof relies on the lemma below, which is an easy consequence of Theorem 6.0.1. We recall that $\tau(n,g)$ is the number of triangulations of genus g with $2n$ faces, and that $\lambda(\theta)$ is the solution of (6.0.2).

Lemma 6.5.1. *Let (g_n) be such that $\frac{g_n}{n} \to \theta$, with $0 \leqslant \theta < \frac{1}{2}$. Then*

$$\frac{\tau(n-1, g_n)}{\tau(n, g_n)} \to \lambda(\theta)$$

Proof. This is a simple consequence of Theorem 6.0.1. Indeed, we have

$$(T_{n,g_n}) \to \mathbb{T}_{\lambda(\theta)}$$

locally. Let t_1 be the triangulation represented on Figure 6.9. We have on the one hand

$$\mathbb{P}\left(t_1 \subset T_{n,g_n}\right) = \frac{\tau_1(n-1, g_n)}{\tau(n, g_n)} = \frac{\tau(n-1, g_n)}{\tau(n, g_n)}$$

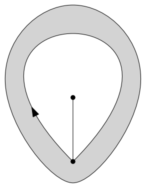


Figure 6.9 – The triangulation t_1 with perimeter 1 and volume 2.

by the usual root transformation (see Remark 6.1.3). On the other hand, we have

$$\mathbb{P}\left(t_1 \subset \mathbb{T}_{\lambda(\theta)}\right) = C_1(\lambda(\theta))\lambda(\theta)^2 = \lambda(\theta).$$

Since $\{t_1 \subset T\}$ is closed and open for d_{loc}, the lemma follows.

□

The idea of the proof of Theorem 6.0.3 is then to write

$$\tau(n, g_n) = \frac{\tau(n, g_n)}{\tau((2+\varepsilon)g_n, g_n)} \times \tau((2+\varepsilon)g_n, g_n) \tag{6.5.1}$$

for some small $\varepsilon > 0$. The first factor can be turned into a telescopic product and estimated by Lemma 6.5.1. The trickier part will be to estimate the second. We will prove the following bounds.

Proposition 6.5.2. *There is a function h with $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that*

$$e^{o(g)} \left(\frac{6}{e}\right)^{2g} (2g)^{2g} \leqslant \tau((2+\varepsilon)g, g) \leqslant e^{h(\varepsilon)g + o(g)} \left(\frac{6}{e}\right)^{2g} (2g)^{2g}$$

as $g \to +\infty$. Moreover, if $\varepsilon_g \to 0$, then

$$\tau((2+\varepsilon_g)g, g) = e^{o(g)} \left(\frac{6}{e}\right)^{2g} (2g)^{2g}.$$

We delay the proof of Proposition 6.5.2, and first explain how to finish the proof of Theorem 6.0.3.

Proof of Theorem 6.0.3. We first note that $f(1/2) = \frac{6}{e}$, so the case $\theta = 1/2$ is exactly the second part of Proposition 6.5.2. We now assume $0 \leqslant \theta < 1/2$. Let $\varepsilon > 0$ be such that $\theta < \frac{1}{2+\varepsilon}$. We estimate the first factor of (6.5.1). We write

$$\frac{1}{n} \log\left(\frac{\tau(n, g_n)}{\tau((2+\varepsilon)g_n, g_n)}\right) = \frac{1}{n} \sum_{i=(2+\varepsilon)g_n}^{n} \log\left(\frac{\tau(i, g_n)}{\tau(i-1, g_n)}\right).$$

By Lemma 6.5.1, when $\frac{g_n}{i} \to \frac{1}{t}$, so $\log \frac{\tau(i,g_n)}{\tau(i-1,g_n)} \to \log \frac{1}{\lambda(1/t)}$. Moreover, by the bounded ratio lemma (Lemma 6.2.1), each of the terms is bounded by some constant C_ε. Hence, by dominated convergence, we have

$$\frac{1}{g_n} \log \left(\frac{\tau(n,g_n)}{\tau((2+\varepsilon)g_n, g_n)} \right) \xrightarrow[n\to+\infty]{} \int_{(2+\varepsilon)}^{1/\theta} \log \frac{1}{\lambda(1/t)} \mathrm{d}t. \tag{6.5.2}$$

On the other hand, if we replace g by g_n with $\frac{g_n}{n} \to \theta$, then the left-hand side of Proposition 6.5.2 becomes $n^{2g_n} \exp\left(\left(2\theta \log \frac{12\theta}{e}\right) n + o(n)\right)$, so Proposition 6.5.2 gives

$$\left(2\theta \log \frac{12\theta}{e}\right) n + o(n) \leqslant \log \frac{\tau((2+\varepsilon)g_n, g_n)}{n^{2g_n}} \leqslant \left(2\theta \log \frac{12\theta}{e} + h(\varepsilon)\right) n + o(n),$$

where $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. By combining this with (6.5.1) and (6.5.2) and finally letting $\varepsilon \to 0$, we get the result. $\square$

We finally prove Proposition 6.5.2. The lower bound can be deduced easily from the Goulden–Jackson formula. For the upper bound, we will bound crudely the number of triangulations by the number of tree-rooted triangulations, i.e. triangulations decorated with a spanning tree. These can be counted by adapting classical operations going back to [Mul67] in the planar case.

Proof of Proposition 6.5.2. We start with the lower bound. The Goulden–Jackson formula (6.1.1) implies the following (crude) inequality:

$$\tau(n,g) \geqslant (36 + o(1)) n^2 \tau(n-2, g-1),$$

where the o is uniform in g, when $n \to +\infty$. By an easy induction, we have

$$\tau((2+\varepsilon)g, g) \geqslant (36+o(1))^g \frac{((2+\varepsilon)g)!}{(\varepsilon g)!} \tau(\varepsilon g, 0) \geqslant (36+o(1))^g \frac{((2+\varepsilon)g)!}{(\varepsilon g)!},$$

and the lower bound follows from the Stirling formula.

For the upper bound, we adapt a classical argument about tree-rooted maps. We denote by $\widetilde{\mathcal{T}}((2+\varepsilon)g, g)$ the set of triangulations of genus g with $2(2+\varepsilon)g$ faces and a distinguished spanning tree, and by $\widetilde{\tau}((2+\varepsilon)g, g)$ its cardinality. We recall that such triangulations have $\varepsilon g + 2$ vertices, so the spanning tree has $\varepsilon g + 1$ edges.

If $t \in \widetilde{\mathcal{T}}((2+\varepsilon)g, g)$, we consider the dual cubic map of t and "cut in two" the edges that are crossed by the spanning tree, as on Figure 6.10. The

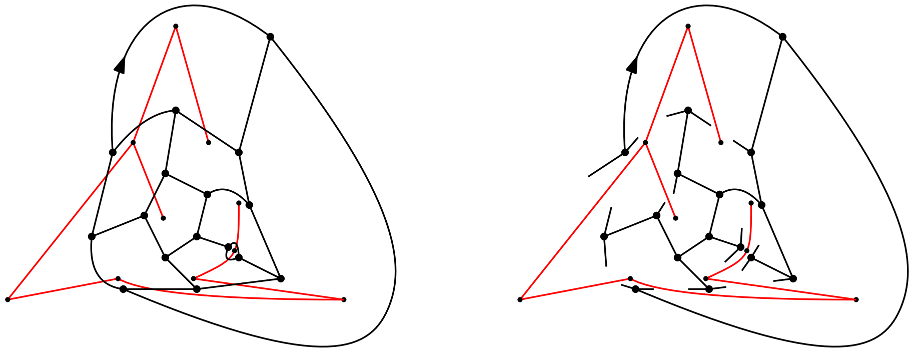


Figure 6.10 – An example of the cutting operation on a cubic map. The distinguished spanning tree is in red. We picked a planar example but in the general case the "opened" map on the right is unicellular.

map that we obtain is unicellular (i.e. it has one face), and we denote it by $U(t)$. Moreover, it is *precubic* (i.e. its vertices have only degree 1 or 3), has genus g and $(6+4\varepsilon)g+1$ edges (the number of edges of the original triangulation, plus one for each edge of the spanning tree). The number of precubic unicellular maps with fixed genus and number of edges is computed exactly in [Cha11, Corollary 7] and is equal to

$$\frac{2((6+4\varepsilon)g+1)!}{12^g g!(2\varepsilon g+2)!((3+2\varepsilon)g)!} = e^{h_U(\varepsilon)g+o(g)} \left(\frac{6}{e}\right)^{2g} (2g)^{2g},$$

where

$$h_U(\varepsilon) = 2\varepsilon \log \frac{6}{\varepsilon} + (3+2\varepsilon)\log\left(1+\frac{2\varepsilon}{3}\right) \xrightarrow[\varepsilon\to 0]{} 0$$

by the Stirling formula. Finally, to go back from $U(t)$ to t, we also need to remember how to match the leaves two by two in the face of $U(t)$ without any crossing. The number of ways to do so is Catalan$(\varepsilon g+1) \leqslant 4^{\varepsilon g}$, so

$$\tau((2+\varepsilon)g,g) \leqslant \tilde{\tau}((2+\varepsilon)g,g) \leqslant 4^{\varepsilon g} e^{h_U(\varepsilon)g+o(g)} \left(\frac{6}{e}\right)^{2g} (2g)^{2g},$$

which is enough to conclude. The proof for the second part of the proposition is exactly the same, but where we replace ε by $\varepsilon_g \to 0$. □

Remark 6.5.3. *Bounding the number of triangulations by the number of tree-rooted triangulations may seem very crude. The reason why this is sufficient is that the spanning trees have only $\varepsilon g+1$ edges, so the number of spanning trees of a triangulation can be bounded by $\binom{3(2+\varepsilon)g}{\varepsilon g+1}$, which is of the form $e^{h(\varepsilon)g+o(g)}$ with $h(\varepsilon)\to 0$ as $\varepsilon \to 0$.*

Chapter 7

Universality of high genus local limits

Abstract. This chapter is adapted from the article *Universality for local limits of high genus maps* with T. Budzinski, in preparation [BL20]. We study a wide class of maps, namely bipartite maps with prescribed degrees, and we prove that they converge locally in distribution to a family of infinite bipartite maps of the plane who share a lot of geometric properties with the PSHT (the local limit of high genus triangulations). This is a result of universality, as we observe similar phenomena for different models of high genus maps. The outline of the proof is similar to what happened with triangulations, showing that the methods developed in the previous chapter are robust. However, everything is more technical and each intermediate result is harder to establish.

Note: this chapter is adapted from a paper in preparation. Therefore, there is a higher chance of minor errors appearing. We only include a sketch of proof for Lemma 7.4.8.

After showing that the local limits of high genus triangulations are the PHSTs, the natural question is to know whether there is a universal phenomenon underlying this convergence. It turns out that it is the case, in this chapter, we tackle the same question as in Chapter 6, but for a much wider class of maps: bipartite maps with prescribed degrees. Instead of having only triangular faces, this time, for all i, we prescribe the (asymptotic) proportion of faces of half-perimeter i. In the planar case, the local convergence was established by Stephenson [Ste18], the peeling process was studied by Budd [Bud15], and several geometric properties by Budd and Curien [BC17].

We prove that the local limit is the Infinite Boltzmann bipartite planar map (abbreviated as IBMP), as defined in Chapter 3 (see also [Bud18a, Appendix C]). This time, the limiting objects are described by an infinity of numbers (a hyperbolicity parameter, as in the PSHT, plus the proportion of faces of each size).

Our result, which applies to a very large class of maps, is a step towards establishing *universality* in the domain of high genus maps, meaning that regardless of the precise model of maps, the same phenomena are observed.

More precisely, we prove the following theorem:

Theorem 7.0.1. *Let* (g_n) *be a sequence such that* $\frac{g_n}{n} \to \theta$ *with* $\theta \in \left[0, \frac{1}{2}\right)$, *and* $\mathbf{f}^{(n)} = (f_1^{(n)}, f_2^{(n)}, \ldots)$ *such that* $\sum_{i \geqslant 1} i f_i^{(n)} = n$. *Suppose that for all* i *there exists an* α_i *such that* $\frac{f_i^{(n)}}{n} \to \alpha_i$, *with the condition that* $\sum_i i\alpha_i = 1$ *and* $\sum_i i^2 \alpha_i < \infty$. *Let* $\mathbf{M}_n$ *be a uniform bipartite map of genus* g_n *whose face degrees are given by* $\mathbf{f}^{(n)}$. *Then*

$$\mathbf{M}_n \xrightarrow[n \to +\infty]{(d)} \mathbb{M}_{\mathbf{q}}$$

for the local topology, where the parameters of $\mathbb{M}_{\mathbf{q}}$ *are deterministic, injective functions of* θ *and the* α_i*'s.*

Note that, in order to have local convergence, it is necessary to have

$$\sum_i i\alpha_i = 1$$

which is equivalent to

$$\sum_{i > M} i f_i^{(n)} \to 0 \tag{7.0.1}$$

as $M \to \infty$ uniformly in n, which is equivalent to saying that the root face is asymptotically almost surely of finite degree. For technical reasons, we prove the theorem with slightly stricter conditions (they are equivalent to saying that the expected degree of the root face is finite). In fact, most of the proof is valid for the most general setting (under conditions (7.0.1)), except for the "two-holes argument" (Section 7.4.1). In the case $\theta = 0$, where the genus is sublinear, this argument is not needed, and therefore the local convergence holds under conditions (7.0.1).

Thanks to Theorem 7.0.1, we can find the asymptotic number of high genus bipartite maps.

Theorem 7.0.2. *Let (g_n) and $\mathbf{f}^{(n)}$ defined as in Theorem 7.0.1. Let $\beta_g(\mathbf{f})$ be the number of bipartite maps of genus g whose face degrees are given by $\mathbf{f}$. Then*

$$\beta_{g_n}(\mathbf{f}^{(n)}) = n^{2g_n} \exp\left(f(\theta, (\alpha_i)_{i\in\mathbb{N}})n + o(n)\right)$$

where f is a function defined in Section 7.5.

The methods developed in Chapter 6 are robust, and therefore the outline of the proof is similar: first we prove the a tightness result, then establish that every potential limit has to be an IBPM with (possibly) random parameters, and finally we prove that these parameters have to be deterministic, therefore establish the result.

However, the key tools of the proof are harder to establish than in Chapter 6, mainly because there are now an infinite number of parameters (instead of one), and that the faces do not have bounded degree. The end of the proof of the local convergence is completely different compared to the case of triangulations (this time, we are not able to compute exactly the inverse degree of the root vertex in the IBPM). Also, the proofs of the bounded ratio lemma, the two-holes argument and the asymptotics are way more technical this time.

Structure of the chapter. In Section 7.1, we review basic definitions and previous results that will be used in all the paper. In Section 7.2, we prove that the maps $\mathbf{M}_n$ are tight for the local topology, and that any subsequential limit is a.s. planar and one-ended. In Section 7.3, we prove Theorem 7.3.1, which implies that any subsequential limit of $\mathbf{M}_n$ is an IBPM with random parameters. In Section 7.4, we conclude the proof of Theorem 7.0.1 by showing that the parameters are deterministic and depends only on θ and the α_i's. Finally, Section 7.5 is devoted to the proof of Theorem 7.0.2, and Section 7.6 contains the proofs of some technical lemmas.

Index of notations

In general, we will we will use lower case letters such as m to denote deterministic objects or quantities, upper case letters such as M for random objects and `mathcal` letters such as $\mathcal{M}$ for sets of objects. We will use `mathbf` characters such as $\mathbf{q}$ for sequences, and normal characters such as q_j for their terms.

- $\mathbf{f} = (f_j)_{j\geqslant 1}$: denotes a face degree sequence.

- g: will denote the genus.
- θ: limit value of $\frac{g}{|\mathbf{f}|}$ when $|\mathbf{f}| \to +\infty$.
- $\mathcal{B}_g(\mathbf{f})$: set of finite bipartite maps with genus g and f_j faces of degree $2j$ for all $j \geqslant 1$.
- $\beta_g(\mathbf{f})$: cardinality of $\mathcal{B}_g(\mathbf{f})$.
- $|\mathbf{f}| = \sum_{j \geqslant 1} j f_j$ (i.e. the number of edges of a map in $\mathcal{B}_g(\mathbf{f})$).
- $\mathbf{q} = (q_j)_{j \geqslant 1}$: denotes a weight sequence ($\mathbf{Q}$ denotes a random weight sequence).
- $\mathbb{M}_{\mathbf{q}}$: the infinite Boltzmann planar map with weight sequence $\mathbf{q}$.
- $W_p(\mathbf{q})$: partition function of finite Boltzmann bipartite maps of the $2p$-gon with weights $\mathbf{q}$.
- $\mathcal{Q}^* = \left\{\mathbf{q} = (q_j)_{j \geqslant 1} \in [0,1]^{\mathbb{N}^*} | \exists j \geqslant 2, q_j > 0\right\}$.
- $\mathcal{Q}_a$: set of admissible families of Boltzmann weights $\mathbf{q}$, i.e. such that $W_p(\mathbf{q})$ converges.
- $\mathcal{Q}_h$: set of Boltzmann weights for which $\mathbb{M}_{\mathbf{q}}$ exists. We have
$$\mathcal{Q}_h \subset \mathcal{Q}_a \cap \mathcal{Q}^*.$$
- $\mathcal{Q}_f$: set of Boltzmann weights for which the expectation of the root face of $\mathbb{M}_{\mathbf{q}}$ is finite.
- $g_{\mathbf{q}}$: for $\mathbf{q} \in \mathcal{Q}_a$, denotes the solution of the equation
$$\sum_{j \geqslant 1} q_j \frac{1}{4^{j-1}} \binom{2j-1}{j-1} g_{\mathbf{q}}^{j-1} = 1 - \frac{4}{g_{\mathbf{q}}}.$$
- $\nu_{\mathbf{q}}(i) = \begin{cases} q_{i+1}\, g_{\mathbf{q}}^i & \text{if } i \geqslant 0, \\ 2W_{-1-i}(\mathbf{q})\, g_{\mathbf{q}}^i & \text{if } i \leqslant -1. \end{cases}$, i.e. $\nu_{\mathbf{q}}$ is the step distribution of the random walk on $\mathbb{Z}$ associated to the peeling process of subcritical or critical $\mathbf{q}$-Boltzmann maps.
- $\omega_{\mathbf{q}} > 1$: for $\mathbf{q} \in \mathcal{Q}_h$, denotes the solution (other than 1) of
$$\sum_{i \in \mathbb{Z}} \omega^i \nu_{\mathbf{q}}(i) = 1.$$

- $C_p(\mathbf{q})$: for $\mathbf{q} \in \mathcal{Q}_h$ and $p \geqslant 1$, constants such that
$$\mathbb{P}\left(m \subset \mathbb{M}_{\mathbf{q}}\right) = C_p(\mathbf{q}) \times \prod_{f \in m} q_{\deg(f)/2}$$
for every finite map m with one hole of perimeter $2p$.

- $a_j(\mathbf{q}) = \frac{1}{j}\mathbb{P}\left(\text{the degree of the root face of } \mathbb{M}_{\mathbf{q}} \text{ is } 2j\right)$ (for all $j \geqslant 1$).

- α_j: denotes a possible value of $(a_j(\mathbf{q}))$, or the limit of the ratio $\frac{f_j}{|\mathbf{f}|}$ when $|\mathbf{f}| \to +\infty$. We will have $\sum_{j \geqslant 1} j\alpha_j = 1$.

- $d(\mathbf{q}) = \mathbb{E}\left[\left(\text{degree of the root vertex in } \mathbb{M}_{\mathbf{q}}\right)^{-1}\right]$.

- $r_j(\mathbf{q}) = \left(g_{\mathbf{q}}\omega_{\mathbf{q}}\right)^{j-1} q_j = \lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}_{P_{i+1}-P_i=j-1}$ for $\mathbf{q} \in \mathcal{Q}_h$ and $j \geqslant 1$, where P is the perimeter process associated to a peeling exploration of $\mathbb{M}_{\mathbf{q}}$ (see Proposition 7.1.2).

- $r_\infty(\mathbf{q}) = \frac{\left(\sqrt{\omega_{\mathbf{q}}}-\sqrt{\omega_{\mathbf{q}}-1}\right)^2}{2\sqrt{\omega_{\mathbf{q}}(\omega_{\mathbf{q}}-1)}} = \lim_{n\to+\infty} \frac{V_n-2P_n}{n}$ for $\mathbf{q} \in \mathcal{Q}_h$ and $j \geqslant 1$, where P and V are the perimeter and volume processes associated to a peeling exploration of $\mathbb{M}_{\mathbf{q}}$ (see Proposition 7.1.2).

- $\mathcal{A}$: denotes a peeling algorithm.

- $\mathcal{E}_n^{\mathcal{A}}(m)$: explored map after n peeling steps on the map m using algorithm $\mathcal{A}$. This is a finite map with holes.

7.1 Preliminaries

Here we recall some definitions and properties from Chapter 3, and introduce some other notions that will be useful in the proof.

7.1.1 Definitions and combinatorics

For every $\mathbf{f} = (f_1, f_2, \ldots)$ and $g \geqslant 0$, we will denote by $\mathcal{B}_g(\mathbf{f})$ the set of bipartite maps of genus g with f_i faces of degree $2i$ for all i. A map of $\mathcal{B}_g(\mathbf{f})$ has $\sum_{i \geqslant 1} if_i$ edges, $\sum_{i \geqslant 1} f_i$ faces and $2-2g+\sum_{i \geqslant 1}(i-1)f_i$ vertices. We will also denote by $\beta_g(\mathbf{f})$ the cardinality of $\mathcal{B}_g(\mathbf{f})$.

Maps with boundaries. We will also need to consider two different notions of bipartite maps with boundaries, that we call *maps with holes* and *maps of multi-polygons*. Basically, the first ones will be used to describe a neighbourhood of the root in a map, and the second ones to describe the complement of this neighbourhood.

For $\ell \geqslant 1$ and $p_1, p_2, \dots, p_\ell \geqslant 1$, we call a *map with holes of perimeter* $2p_1, \dots, 2p_\ell$ a map with special marked faces (called *holes*), degree $2p_i$ for every $1 \leqslant i \leqslant \ell$. The faces h_i are called the *holes*. Maps with holes will always be finite, and the root might be anywhere in the map. *Maps of the* $(2p_1, \dots, 2p_\ell)$*-gon* have a similar definition, except that they may be infinite, and this time they have ℓ distinct roots, one on each of the boundaries. Note that, contrary to triangulations, the boundaries are not necessarily simple.

Let $\mathcal{B}_g^{(p_1,p_2,\dots,p_\ell)}(\mathbf{f})$ (resp. $\widetilde{\mathcal{B}}_g^{(p_1,p_2,\dots,p_\ell)}(\mathbf{f})$) the set of bipartite maps of the $(2p_1, 2p_2, \dots, 2p_\ell)$-gon (resp. with holes of sizes $(2p_1, 2p_2, \dots, 2p_\ell)$) of genus g with interior faces given by $\mathbf{f}$ and $\beta_g^{(p_1,p_2,\dots,p_\ell)}(\mathbf{f})$ (resp. $\widetilde{\beta}_g^{(p_1,p_2,\dots,p_\ell)}(\mathbf{f})$) its cardinality.

Map inclusion. Given a map M, let M^* be its dual map. Let $\mathfrak{e}$ be a connected subset of edges of M^* such that the root vertex of M^* is adjacent to $\mathfrak{e}$. To $\mathfrak{e}$, we associate the map $\mathfrak{m}$ that is obtained by gluing the faces of M corresponding to the vertices adjacent to $\mathfrak{e}$ along the (dual) edges of $\mathfrak{e}$ (see Figure 7.1). Note that $\mathfrak{m}$ is a map with simple holes.

Now, if m is a finite bipartite map with simple holes, and M a (finite or infinite) map, we say that

$$m \subset M$$

if m can be obtained from M by the procedure described above, or if m is the map with only one edge, or if m is the map with a simple boundary of size $2p$ and one face of degree $2p$, where $2p$ is the degree of the root face in M.

Generating functions. First, fix a sequence of variables $\mathbf{q} = (q_1, q_2, \dots)$. To any planar bipartite map m (with or without a boundary), we associate a weight

$$w_{\mathbf{q}}(m) = \prod_{f \in m} q_{\deg f/2}, \tag{7.1.1}$$

where the product spans over all faces of m, not including the boundary. The generating function[1] of planar bipartite maps

$$W(\mathbf{q}) = \sum_m w_{\mathbf{q}}(m)$$

[1] also called series, or partition function.

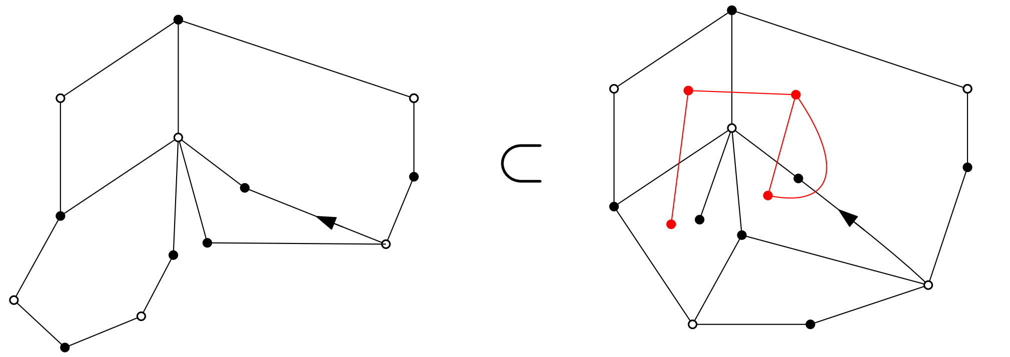


Figure 7.1 – Inclusion of bipartite maps, on an example.

where the sum spans over all planar bipartite maps m. We can define a similar partition function $W_p(\mathbf{q})$ for maps of the $2p$-gon:

$$W_p(\mathbf{q}) = \sum_m w_{\mathbf{q}}(m)$$

where the sum spans over all planar bipartite maps m of the $2p$-gon. Note that $W_1(\mathbf{q}) = W(\mathbf{q})$.

Given a sequence of positive reals $\mathbf{q}$, let

$$f_{\mathbf{q}}(x) = \sum_{k \geqslant 1} q_k \binom{2k-1}{k-1} x^{k-1}.$$

If the equation

$$f_{\mathbf{q}}(x) = 1 - \frac{1}{x} \tag{7.1.2}$$

has a positive solution Z_q, then $\mathbf{q}$ is said to be *admissible*. For all p, $W_p(\mathbf{q})$ converges if and only if $\mathbf{q}$ is admissible [Cur19].

The *Boltzmann distribution* of parameter $\mathbf{q}$ on finite planar bipartite maps of the $2p$-gon is a measure such that

$$\mathbb{P}(m) = \frac{w_{\mathbf{q}}(m)}{W_p(\mathbf{q})}$$

for all m.

7.1.2 A counting formula

As in Chapter 6, we do not have a priori precise asymptotics, once again we rely on a recurrence formula (proved in Chapter 5):

$$\binom{n+1}{2}\beta_g(\mathbf{f}) = \sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ g_1+g_2+g^*=g}} (1+n_1)\binom{v_2}{2g^*+2}\beta_{g_1}(\mathbf{s})\beta_{g_2}(\mathbf{t}) + \sum_{g^*\geqslant 0}\binom{v+2g^*}{2g^*+2}\beta_{g-g^*}(\mathbf{f}) \tag{7.1.3}$$

where $n=\sum_i if_i$, $n_1=\sum_i is_i$, $v=2-2g+n-\sum_i f_i$, $v_2=2-2g_2+n_2-\sum_i t_i$ and $n_2=\sum_i it_i$ (the n's count edges, the v's count vertices, in accordance with the Euler formula), with the convention that $\beta_g(\mathbf{0})=0$. This formula will play the same role as the formula for triangulations [GJ08] in Chapter 6.

7.1.3 The lazy peeling process of bipartite maps

Here we recall the *peeling process* from Chapter 3. The *peeling algorithm* is a function $\mathcal{A}$ that takes as input a finite planar bipartite map m with boundaries, and that outputs an edge $\mathcal{A}(m)$ on the boundary of m.

Given an infinite planar bipartite map M and a peeling algorithm $\mathcal{A}$, we can define an increasing sequence $\left(\mathcal{E}_M^{\mathcal{A}}(k)\right)_{k\geqslant 0}$ of maps with holes such that $\mathcal{E}_M^{\mathcal{A}}(k)\subset M$ for every k in the following way. First, the map $\mathcal{E}_M^{\mathcal{A}}(0)$ is the trivial map consisting of the root edge only. For every $k\geqslant 1$, let F_k be the face (of M) on the other side of $\mathcal{A}(\mathcal{E}_M^{\mathcal{A}}(k))$. There are two possible cases (see Figure 7.2):

- either F_k doesn't belong to $\mathcal{E}_M^{\mathcal{A}}(k)$, and $\mathcal{E}_M^{\mathcal{A}}(k+1)$ is the map obtained from $\mathcal{E}_M^{\mathcal{A}}(k)$ by gluing a simple face of size $\deg F_k$ to $\mathcal{A}(\mathcal{E}_M^{\mathcal{A}}(k))$,
- or F_k belongs to $\mathcal{E}_M^{\mathcal{A}}(k)$. In that case, it means that there exists an edge e_k on the same boundary as $\mathcal{A}(\mathcal{E}_M^{\mathcal{A}}(k))$ such that those two edges are actually identified in M. The map $\mathcal{E}_M^{\mathcal{A}}(k+1)$ is obtained from $\mathcal{E}_M^{\mathcal{A}}(k)$ by gluing $\mathcal{E}_M^{\mathcal{A}}(k)$ and e_k together, and if this creates a finite hole, by filling it (such that the resulting map is included in M, there is only one possible way of doing so).

Such an exploration is called *filled-in*, because all the finite holes are filled at each step.

7.1.4 Infinite Boltzmann bipartite planar maps

Definitions. Our goal here is to recall the definition of infinite Boltzmann bipartite planar maps. We will recall results from [Bud15] in the critical case

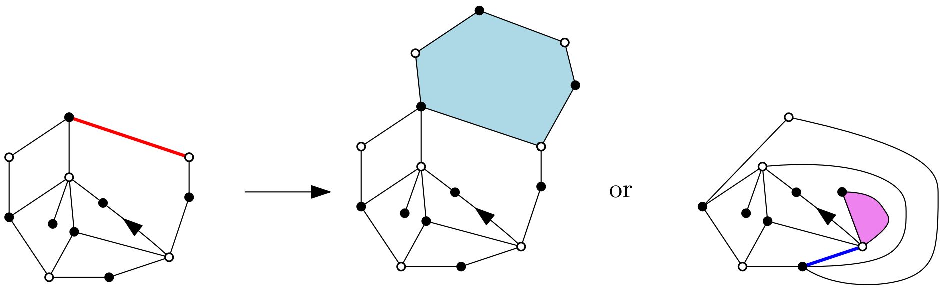


Figure 7.2 – The lazy peeling on an example. The chosen edge is in red. Either a new face is discovered (center), or the chosen edge is glued to another edge of the boundary (right, in blue).

(i.e. maps that occur as local limits when the genus is small compared to the size), and from [Bud18a, Appendix C] and [Cur19] in the subcritical case (i.e. maps that are expected to be the limits when the genus is proportional to the size).

Let $\mathbf{q} = (q_i)_{i \geqslant 1}$ be a sequence of nonnegative real numbers that we will call the *Boltzmann weights*. A random infinite bipartite planar map M is called $\mathbf{q}$-Boltzmann if there are constants $(C_p(\mathbf{q}))_{p \geqslant 1}$ if, for every finite bipartite map m with one hole of half-perimeter p, we have

$$\mathbb{P}\left(m \subset M\right) = C_p(\mathbf{q}) w_{\mathbf{q}}(m).$$

We will see that given $\mathbf{q}$, such a map does not always exist, but when it does, it is unique, i.e. the constants $C_p(\mathbf{q})$ are determined by $\mathbf{q}$. More precisely, as noted in [Bud18a, Appendix C], if a $\mathbf{q}$-Boltzmann map exists, then the partition function of maps of a 2-gon with Boltzmann weights $\mathbf{q}$ must be finite, which is equivalent to the admissibility criterion (7.1.2). Moreover, with the notation of Section 7.1.1, we call $\mathbf{q}$ *critical* if $f'_{\mathbf{q}}(Z_{\mathbf{q}}) = \frac{1}{Z_{\mathbf{q}}^2}$ and *subcritical* if this is not the case.

Finally, we define a measure $\nu_{\mathbf{q}}$ on $\mathbb{Z}$ as follows:

$$\nu_{\mathbf{q}}(i) = \begin{cases} q_{i+1}\, g_{\mathbf{q}}^i & \text{if } i \geqslant 0, \\ 2W_{-1-i}(\mathbf{q})\, g_{\mathbf{q}}^i & \text{if } i \leqslant -1. \end{cases} \tag{7.1.4}$$

As noted in [Bud15], this is the step distribution of the random walk on $\mathbb{Z}$ describing the evolution of the perimeter of a finite $\mathbf{q}$-Boltzmann map with a large perimeter (see also [Cur19, Chapter 5.1]).

Theorem 7.1.1. *1. If a $\mathbf{q}$-Boltzmann infinite map exists, it is unique (in distribution), so we can denote it by $\mathbb{M}_{\mathbf{q}}$ if it exists.*

2. *If* $\mathbf{q}$ *is not admissible, then* $\mathbb{M}_{\mathbf{q}}$ *does not exist.*

3. *If* $\mathbf{q}$ *is admissible and critical, then* $\mathbb{M}_{\mathbf{q}}$ *exists.*

4. *If* $\mathbf{q}$ *is admissible and sub-critical, then* $\mathbb{M}_{\mathbf{q}}$ *exists if and only if the equation*
$$\sum_{i\in\mathbb{Z}} \nu_{\mathbf{q}}(i)\omega^i = 1 \tag{7.1.5}$$
has a solution $\omega > 1$.

5. *In this case, the solution* ω *is unique and, for every* $p \geqslant 1$, *we have*
$$C_p(\mathbf{q}) = (g_{\mathbf{q}}\omega)^{p-1} \sum_{i=0}^{p-1} (4\omega)^{-i} \binom{2i}{i}. \tag{7.1.6}$$

The third point is from [Bud15], and the others are from [Bud18a, Appendix C] [2]. When it exists, we will call the map $\mathbb{M}_{\mathbf{q}}$ the $\mathbf{q}$-*IBPM* (for *Infinite Boltzmann Planar Map).*

The random walk $\widetilde{\nu}_{\mathbf{q}}$. To study the $\mathbf{q}$-IBPM, we define the measure $\widetilde{\nu}_{\mathbf{q}}$ on $\mathbb{Z}$ by $\widetilde{\nu}_{\mathbf{q}}(i) = \omega^i \nu_{\mathbf{q}}(i)$, where ω is given by (7.1.5) if $\mathbf{q}$ is subcritical, and $\omega = 1$ if $\mathbf{q}$ is critical. The random walk with step distribution $\widetilde{\nu}_{\mathbf{q}}$ plays an important role when studying $\mathbb{M}_{\mathbf{q}}$. We first note that this walk has a positive drift. Indeed, we have
$$\sum_{i\in\mathbb{Z}} i\widetilde{\nu}_{\mathbf{q}}(i) = F'_{\mathbf{q}}(\omega) > 0,$$
since $F_{\mathbf{q}}$ is convex and takes the value 1 at 1 and $\omega > 1$. Note also that it is possible that the drift is $+\infty$.

Lazy peeling explorations of the q-IBPM. We now perform a few computations related to the lazy peeling exploration of the $\mathbf{q}$-IBPM. For this, we fix a peeling algorithm $\mathcal{A}$, and consider an filled-in exploration of $\mathbb{M}_{\mathbf{q}}$ according to $\mathcal{A}$ (that is, at each step, if a finite region is separated from infinity, we discover it completely). We recall that $\mathcal{E}_n^{\mathcal{A}}(\mathbb{M}_{\mathbf{q}})$ is the explored region at time n, and we denote by $(\mathcal{F}_n)_{n\geqslant 0}$ the filtration generated by this exploration. We denote by P_n (resp. V_n) the half-perimeter (resp. total number of edges) of $\mathcal{E}_n^{\mathcal{A}}(\mathbb{M}_{\mathbf{q}})$. Then it follows from the definition of $\mathbb{M}_{\mathbf{q}}$ is

[2] We correct a small mistake from [Bud18a, Appendix C], where $g_{\mathbf{q}}$ was omitted in the formula for $C_p(\mathbf{q})$.

a Markov chain. More precisely, it is an h-transform of the walk with step distribution $\widetilde{\nu}_{\mathbf{q}}$, i.e. it has the following transitions:

$$\mathbb{P}\left(P_{n+1} = P_n + i | \mathcal{F}_n\right) = \widetilde{\nu}_{\mathbf{q}}(i) \frac{h(P_n + i)}{h(P_n)}, \tag{7.1.7}$$

where $h(p) = \sum_{i=0}^{p-1} \frac{1}{(4\omega)^i} \binom{i}{2i}$. In particular, if $\mathbf{q}$ is critical, then we have $h(p) = \frac{2p}{4^p}\binom{2p}{p} \sim \frac{2}{\sqrt{\pi}}\sqrt{p}$. If not, then $\omega > 1$ and $h(p) \to \sqrt{\frac{\omega}{\omega - 1}}$ as $p \to +\infty$.

As noted in [Cur19], this shows that P is an h-transform of the random walk with step distribution $\widetilde{\nu}_{\mathbf{q}}$, and more precisely that P has the distribution of this random walk, conditioned to stay positive.

IBPM with finite expected degree of the root face. We denote by $\mathcal{Q}_f$ the set of admissible weight sequences such that $\mathbb{M}_{\mathbf{q}}$ exists, and the degree of the root face of $\mathbb{M}_{\mathbf{q}}$ has finite expectation. Note that, if $\mathbb{M}_{\mathbf{q}}$ exists, the degree of the root face is determined by the first peeling step, so

$$\mathbb{P}\left(\text{the root face has degree } 2j\right) = \frac{h(j)}{h(1)} \widetilde{\nu}_{\mathbf{q}}(j-1) = \frac{h(j)}{h(1)} \left(g_{\mathbf{q}}\omega\right)^{j-1} q_j \tag{7.1.8}$$

for $j \geqslant 1$. If $\mathbf{q}$ is critical, the right hand-side is equivalent to $\frac{2}{\sqrt{\pi}}\sqrt{j} g_{\mathbf{q}}^{j-1} q_j$ as $j \to +\infty$, so $\mathbf{q} \in \mathcal{Q}_f$ if and only if

$$\sum_{j \geqslant 1} j^{3/2} g_{\mathbf{q}}^j q_j < +\infty. \tag{7.1.9}$$

On the other hand, we recall (see e.g. [Cur19, Chapter 5.2]) that a critical weight sequence $\mathbf{q}$ is called *of type* $\frac{5}{2}$, or *critical generic*, if

$$\sum_{j \geqslant 1} (j-1)(j-2)\binom{2j-1}{j-1} q_j \left(\frac{g_{\mathbf{q}}}{4}\right)^{j-3} < +\infty,$$

which is clearly equivalent to (7.1.9).

In the subcritical case, by (7.1.8), we have

$$\sum_{j \geqslant 1} j \widetilde{\nu}_{\mathbf{q}}(j) < +\infty,$$

i.e. the drift of $\widetilde{\nu}_{\mathbf{q}}$ is finite. To sum up:

- In the critical case, $\mathbf{q} \in \mathcal{Q}_f$ if and only if $\mathbf{q}$ is critical generic, which means that the perimeter process (P_n) converges to a $3/2$-stable Lévy process with no positive jump conditioned to be positive (see [Cur19, Theorem 10.1]). This basically means that $\mathbf{q}$-Boltzmann finite maps lie in the domain of attraction of the Brownian map [MM07].

- In the subcritical case, $\mathbf{q} \in \mathcal{Q}_f$ if and only if the measure $\widetilde{\nu}_{\mathbf{q}}$ has finite expectation, which means that the perimeter P has linear growth (instead of super-linear if the expectation of $\widetilde{\nu}_{\mathbf{q}}$ was infinite).

7.1.5 Three ways to describe Boltzmann weights

Three parametrizations of $\mathcal{Q}_h$ In this work, we will make use of three different "parametrizations" on the set $\mathcal{Q}_h$ of families $\mathbf{q}$ of Boltzmann weights such that $\mathbb{M}_{\mathbf{q}}$ exists. The first one, already described above, is to use directly the Boltzmann weights q_i. It is the simplest one to define the model $\mathbb{M}_{\mathbf{q}}$ and gives the simplest description of its distribution.

The second parametrization we will use is the one given by Proposition 7.1.2: we describe $\mathbf{q}$ by the parameters $r_j(\mathbf{q}) \in [0,1)$ and by $r_\infty(\mathbf{q}) \in (0,+\infty]$. Here $r_j(\mathbf{q})$ describes the proportion of steps where we discover a face of degree $2j$ during a peeling exploration of $\mathbb{M}_{\mathbf{q}}$, and $r_\infty(\mathbf{q})$ compares volume and perimeter growth. The advantage of these parameters is that they allow to "read" $\mathbf{q}$ as an almost sure observable on a peeling exploration of the map $\mathbb{M}_{\mathbf{q}}$. This will be useful in Section 7.4.1.

The third parametrization is to use on the one hand the distribution of the root face, and on the other hand the average degree of the vertices. More precisely, for $i \geqslant 1$, we write

$$a_i(\mathbf{q}) = \frac{1}{i}\mathbb{P}\left(\text{the root face of } \mathbb{M}_{\mathbf{q}} \text{ has degree } 2i\right).$$

We also write $d(\mathbf{q}) = \mathbb{E}\left[\frac{1}{\deg_{\mathbb{M}_{\mathbf{q}}}(\rho)}\right]$, where ρ is the root vertex. The advantage of this parametrization is that analogue of these quantities are easy to compute if we replace $\mathbb{M}_{\mathbf{q}}$ by a finite uniform map with prescribed genus and face degrees. This will be useful in the end of the proof of Theorem 7.0.1. However, it is not obvious at all that $(a_i(\mathbf{q}))_{i \geqslant 1}$ and $d(\mathbf{q})$ are sufficient to characterize $\mathbf{q}$. We will actually prove this for $\mathbf{q} \in \mathcal{Q}_a$ as a consequence of local limit arguments (Proposition 7.4.7).

The goal of this section is to gather a few simple properties of these three parametrizations and the relations between them.

Recovering $\mathbf{q}$ from explorations of $\mathbb{M}_{\mathbf{q}}$. We now describe more precisely our second parametrization of $\mathbf{q}$. More precisely, the next result basically states that we can recover the weight sequence $\mathbf{q}$ by just observing the processes P and V during a peeling exploration of $\mathbb{M}_{\mathbf{q}}$.

Proposition 7.1.2. *We have the following almost sure convergences:*

$$\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{1}_{P_{i+1}-P_i=j-1}\xrightarrow[n\to+\infty]{a.s.}(g_{\mathbf{q}}\omega)^{j-1}q_j=:r_j(\mathbf{q})\tag{7.1.10}$$

for every $j\geqslant 1$, *and*

$$\frac{V_n-2P_n}{n}\xrightarrow[n\to+\infty]{a.s.}\frac{\left(\sqrt{\omega}-\sqrt{\omega-1}\right)^2}{2\sqrt{\omega(\omega-1)}}=:r_\infty(\mathbf{q}).\tag{7.1.11}$$

Moreover, the weight sequence $\mathbf{q}$ *is a measurable function of the numbers* $r_j(\mathbf{q})$ *for* $j\in\mathbb{N}^*\cup\{\infty\}$.

Proof. In the subcritical case, the second convergence is Proposition 10.12 of [Cur19]. In the critical case, we have $\omega=1$ so the right-hand side of (7.1.11) is infinite, and the result follows from Theorem 10.8 of [Cur19].

Let us now prove the first convergence. For this, we note that we have $P_n\to+\infty$ as $n\to+\infty$. In the subcritical case, it also follows from Proposition 10.12 of [Cur19] that (P_n) has a.s. linear growth. In the critical case, this is a consequence of Lemma 10.9 of [Cur19]. On the other hand, given the asymptotics for h right after (7.1.7), for any fixed $j\geqslant 1$, we have $\frac{h(p+j-1)}{h(p)}\to 1$ as $p\to+\infty$. It follows that

$$\mathbb{P}\left(P_{n+1}-P_n=j-1|\mathcal{F}_n\right)\xrightarrow[n\to+\infty]{a.s.}\widetilde{\nu}_{\mathbf{q}}(j-1)=(g_{\mathbf{q}}\omega)^{j-1}q_j,$$

and the first convergence follows by the law of large numbers.

Finally, the function $\omega\to\frac{\left(\sqrt{\omega}-\sqrt{\omega-1}\right)^2}{2\sqrt{\omega(\omega-1)}}$ is a decreasing homeomorphism from $[1,+\infty)$ to $(0,+\infty]$, so ω is a measurable function of $r_\infty(\mathbf{q})$. Moreover, by the definition of $g_{\mathbf{q}}$, we have

$$1-\frac{4}{g_{\mathbf{q}}}=\sum_{j\geqslant 1}\binom{2j-1}{j-1}q_j\left(\frac{g_{\mathbf{q}}}{4}\right)^{j-1}=\sum_{j\geqslant 1}\frac{1}{4^j}\binom{2j-1}{j-1}\frac{r_j(\mathbf{q})}{\omega^{j-1}},$$

which implies that $g_{\mathbf{q}}$ is a measurable function of ω and the numbers $r_j(\mathbf{q})$ for $j\in\mathbb{N}^*$. From here, we easily recover the q_j. $\square$

Weight sequences corresponding to a given distribution of the root face. We know study some basic properties of the third parametrization. The next result shows that the set of infinite Boltzmann maps with a given distribution of the root face is one-parameter, and can be parametrized by $\omega_{\mathbf{q}}$. We recall that for any $\mathbf{q}\in\mathcal{Q}_a$, the numbers $a_i(\mathbf{q})$ satisfy $\sum_{i\geqslant 1}ia_i(\mathbf{q})=1$, and we have $\omega_{\mathbf{q}}\geqslant 1$.

Proposition 7.1.3. *Let (α_i) be such that $\sum_{i \geqslant 1} i\alpha_i = 1$ and $\alpha_1 \neq 1$, and let $\omega \geqslant 1$. Then there is a unique $\mathbf{q} \in \mathcal{Q}_h$ such that*

$$\omega_{\mathbf{q}} = \omega \text{ and } \forall j \geqslant 1,\, a_j(\mathbf{q}) = \alpha_j.$$

Proof. We start with uniqueness. We note that

$$a_j(\mathbf{q}) = \frac{1}{j} C_j(\mathbf{q}) q_j = \frac{1}{j} \left(g_{\mathbf{q}} \omega_{\mathbf{q}}\right)^{j-1} h_{\omega_{\mathbf{q}}}(j) q_j, \tag{7.1.12}$$

so q_j can be obtained as a function of $a_j(\mathbf{q}) = \alpha_j$, $\omega_{\mathbf{q}}$ and $g_{\mathbf{q}}$. Moreover, by the definition of $g_{\mathbf{q}}$, we have

$$1 - \frac{4}{g_{\mathbf{q}}} = \sum_{i \geqslant 1} \frac{1}{4^{i-1}} \binom{2i-1}{i-1} q_i g_{\mathbf{q}}^{i-1} = \sum_{i \geqslant 1} \frac{1}{4^{i-1}} \binom{2i-1}{i-1} \frac{i a_i(\mathbf{q})}{\omega_{\mathbf{q}}^{i-1} h_{\omega_{\mathbf{q}}}}, \tag{7.1.13}$$

so $g_{\mathbf{q}}$ can be deduced from $\omega_{\mathbf{q}}$ and $\left(a_j(\mathbf{q})\right)_{j \geqslant 1}$. More precisely, if $\omega_{\mathbf{q}} = \omega$ and $a_j(\mathbf{q}) = \alpha_j$ for all $j \geqslant 1$, then we must have

$$q_j = \frac{j \alpha_j}{\omega^{j-1} h_j(\omega)} \left(\frac{1 - \sum_{i \geqslant 1} \frac{1}{4^{i-1}} \binom{2i-1}{i-1} \frac{i\alpha_i}{\omega^{i-1} h_i(\omega)}}{4} \right)^{j-1}. \tag{7.1.14}$$

To prove the existence, it is enough to check that, for all $\omega \geqslant 1$ and $(\alpha_j)_{j \geqslant 1}$ with $\sum j\alpha_j = 1$ and $\alpha_1 < 1$, the sequence $\mathbf{q}$ given by (7.1.14) is indeed in $\mathcal{Q}_h$, with $\omega_{\mathbf{q}} = \omega$ and $a_j(\mathbf{q}) = \alpha_j$ for all j. The proof is largely inspired by similar arguments in the critical case (see e.g. [Cur19, Lemma 5.2]).

Following (7.1.13), we first write

$$g = \frac{4}{1 - \sum_{i \geqslant 1} \frac{1}{4^{i-1}} \binom{2i-1}{i-1} \frac{i\alpha_i}{\omega^{i-1} h_\omega(i)}},$$

and check that $\mathbf{q}$ is admissible with $g_{\mathbf{q}} = g$. First $\omega^{i-1} h_\omega(i)$ is a polynomial in ω with nonnegative coefficients so $\omega^{i-1} h_\omega(i) \geqslant h_1(i) = \frac{2i}{4^i}\binom{2i}{i}$. From here, we get

$$\sum_{i \geqslant 1} \frac{1}{4^{i-1}} \binom{2i-1}{i-1} \frac{i\alpha_i}{\omega^{i-1} h_\omega(i)} \leqslant \sum_{i \geqslant 1} \alpha_i < \sum_{i \geqslant 1} i\alpha_i = 1$$

because $\alpha_1 < 1$. Therefore, the numbers q_j are nonnegative and $g > 0$, and we can rewrite (7.1.14) as

$$q_j = \frac{j\alpha_j}{(\omega g)^{j-1} h_\omega(j)}.$$

From here, we get

$$\sum_{i \geqslant 1} \frac{1}{4^{i-1}} \binom{2i-1}{i-1} q_i g^{i-1} = 1 - \frac{4}{g}$$

immediately by the definition of g, which proves $\mathbf{q} \in \mathcal{Q}_h$ and $g_{\mathbf{q}} = g$.

Also, we know that there is $j \geqslant 2$ with $\alpha_j > 0$, which implies $q_j > 0$, so $\mathbf{q} \in \mathcal{Q}^*$. We now prove $\mathbf{q} \in \mathcal{Q}_h$ with $\omega_{\mathbf{q}} = \omega$, which is equivalent to proving

$$\sum_{i \in \mathbb{Z}} \nu_{\mathbf{q}}(i) \omega^i = 1,$$

where we recall that $\nu_{\mathbf{q}}$ is defined by (7.1.4). For this, the basic idea will be to show that $(\omega^i h_\omega(i))_{i \geqslant 1}$ is harmonic for $\nu_{\mathbf{q}}$. More precisely, the fact that $\sum_{i \geqslant 1} i\alpha_i = 1$ can be interpreted as a harmonicity relation at 1:

$$\sum_{i \in \mathbb{Z}} h_\omega(i+1) \omega^i \nu_{\mathbf{q}}(i) = \sum_{j \geqslant 1} \omega^{j-1} h_\omega(j) g^{j-1} q_j = \sum_{j \geqslant 1} j\alpha_j = 1 = h_\omega(1), \quad (7.1.15)$$

where in the beginning we do the change of variables $j = i+1$. On the other hand, we know that $h_\omega(p) = \sum_{i=0}^{p-1} \omega^{-i} u(i)$, where $u(i) = \frac{1}{4^i}\binom{2i}{i}$ for $i \geqslant 0$ and we set thee convention $u(i) = 0$ for $i \leqslant -1$. But the same function u plays an important role in the description of the law of the peeling process of finite Boltzmann maps. In particular, we know that u is $\nu_{\mathbf{q}}$-harmonic on positive integers for any admissible weight sequence $\mathbf{q}$ (this can be found in the proof of Lemma 5.2 in [Cur19]). That is, for all $j \geqslant 1$, we have

$$u(j) = \sum_{i \in \mathbb{Z}} \nu_{\mathbf{q}}(i) u(i+j).$$

Multiplying by ω^{-j} and summing over $1 \leqslant j \leqslant p-1$, we get, for all $p \geqslant 1$:

$$h_\omega(p) - h_\omega(1) = \sum_{i \in \mathbb{Z}} \omega^i \nu_{\mathbf{q}}(i) \left(h_\omega(p+i) - h_\omega(i+1) \right).$$

Summing this with (7.1.15) and dividing by $h_\omega(p)$, we obtain

$$\sum_{i \in \mathbb{Z}} \omega^i \nu_{\mathbf{q}}(i) \frac{h_\omega(p+i)}{h_\omega(p)} = 1.$$

for all $p \geqslant 1$. When $p \to +\infty$, we have that $h_\omega(p)$ has a positive limit if $\omega < 1$ and is equivalent to $\frac{2}{\sqrt{\pi}}$ if $\omega = 1$, so $\frac{h_\omega(p+i)}{h_\omega(p)} \to 1$. Therefore, by dominated convergence, we get

$$\sum_{i \geqslant 1} \nu_{\mathbf{q}}(i) \omega^i = 1,$$

where the domination is immediate for negative values of i, and comes from the convergence of the sum (7.1.15) for positive values of i. This proves $\mathbf{q} \in \mathcal{Q}_h$ with $\omega_{\mathbf{q}} = \omega$, and from here $a_j(\mathbf{q}) = \alpha_j$ is immediate using (7.1.12). □

We conclude with a result stating that we can recover $\mathbf{q}$ from the law of the root face degree and a single weight q_j, provided $j \geqslant 2$. More precisely, if we fix $(\alpha_j)_{j\geqslant 1}$ satisfying the assumptions of Proposition 7.1.3, we can denote by $\mathbf{q}^{(\omega)}$ the unique weight sequence for which the law of the root face is given by $(\alpha_j)_{j\geqslant 1}$ and $\omega_{\mathbf{q}^{(\omega)}} = \omega$.

Lemma 7.1.4. *For every $j \geqslant 1$, the function $\omega \to q_j^{(\omega)}$ is nonincreasing. Moreover, if $j \geqslant 2$, this function is decreasing.*

The proof of this lemma is given in the appendix.

7.1.6 Local convergence and dual local convergence

As in the planar case, to define the local convergence, we first need to define balls of bipartite maps. Let m be a finite map. As usual, for every $r \geqslant 1$, we denote by $B_r(m)$ the submap of m spanned by all edges which have an endpoint at distance at most $r-1$ of the root vertex (i.e. consisting of all the faces adjacent to these edges, together with the vertices and edges adjacent to these faces).

We denote by $\partial B_r(m)$ the boundary of $B_r(m)$, i.e. the set of edges e such that exactly one side of e is adjacent to a face of $B_r(m)$. The map $B_r(m)$ is a finite map with holes. We also write $B_0(m)$ for the trivial bipartite map consisting of only one edge.

For any two finite maps m and m', we write

$$d_{\text{loc}}(m, m') = \left(1 + \max\{r \geqslant 0 | B_r(m) = B_r(m')\}\right)^{-1}.$$

This is the *local distance* on the set of finite bipartite maps. As in the planar case, its completion $\overline{\mathcal{B}}$ is a Polish space, which can be viewed as the set of (finite or infinite) bipartite maps in which all the vertices have finite degree. However, this space is not compact.

We also introduce the *dual local convergence*: d^* is the graph distance on the *dual* of a bipartite map. For any finite map m and any $r \geqslant 0$, we denote by $B_r^*(m)$ the map formed by all the faces at dual distance at most r from the root face, along with all their vertices and edges. Like $B_r(m)$, this is a finite map with holes. For any two finite triangulations m and m', we write

$$d^*_{\text{loc}}(t, t') = \left(1 + \max\{r \geqslant 0 | B^*_r(t) = B^*_r(t')\}\right)^{-1}.$$

Lemma 7.1.5. *Let $\mathbf{f}^{(n)}$ such that $\sum_{i>M} i f_i^{(n)} \to 0$ as $M \to \infty$ uniformly in n, then for all sequences g_n and m_n a uniform map in $\mathcal{B}_{g_n}(\mathbf{f}^{(n)})$. Then the sequence (m_n) is tight for d^*_{loc}*

Proof. Let m_n^* be the dual map of (m_n). Then the degree of the root of m_n^* is a.s. finite, and the law of m_n^* is invariant by rerooting, by the classical argument of [AS03, Lemma 4.4], the sequence (m_n^*) is tight for the local topology. □

Roughly speaking, the main steps of our proof for tightness will be the following. By the above Lemma, the sequence (m_n) is tight for d^*_{loc}. We will prove that every subsequential limit is planar and one-ended, and finally that its vertices must have finite degree. As in Chapter 6, we have the following Lemma that allows us to conclude (the proof is the same).

Lemma 7.1.6. *Let (m_n) be a sequence of maps of $\overline{\mathcal{B}}$. Assume that*

$$m_n \xrightarrow[n\to+\infty]{d^*_{\text{loc}}} m,$$

with $m \in \overline{\mathcal{B}}$. Then $m_n \to m$ for d_{loc} when $n \to +\infty$.

7.2 Tightness

The goal of this section is to prove the following result:

Proposition 7.2.1. *The sequence $(\mathbf{M}_n)$ is tight for d_{loc}.*

Note that this results actually holds in a more general setting than Theorem 7.0.1. Indeed, only condition (7.0.1) is required.

7.2.1 The Bounded Ratio Lemma

Fix some numbers $\kappa, \delta > 0$ and a function $A : (0, 1] \to \mathbb{N}$. Let g, n be integers and $(f_1, f_2, \ldots)$ a sequence of numbers such that

$$\sum_i i f_i = n$$

and

$$v := \sum_i (i-1) f_i + 2 - 2g > \kappa n, \tag{7.2.1}$$

and that for all $\varepsilon > 0$
$$\sum_{i<A(\varepsilon)} if_i > (1-\varepsilon)n. \tag{7.2.2}$$

Let i such that $if_i > \delta n$. Then there exists a constant C depending only on δ, κ and the function A such that

Lemma 7.2.2 (Bounded ratio lemma)**.**
$$\beta_g(\mathbf{f}-\mathbf{1}_i) > C\beta_g(\mathbf{f}).$$

Remark 7.2.3. *Note that the conditions taken in this section are less restrictive than those of Theorem 7.0.1, and therefore Lemma 7.2.2 holds in a more general setting.*

Before getting to the proof, we mention an immediate corollary that will be useful later.

Lemma 7.2.4. *Under the same assumptions as in the previous lemma, for all (p,p'), there exists a constant $\widetilde{C}$ depending only on C such that:*

- $\widetilde{\beta}_g^{(p)}(\mathbf{f}-\mathbf{1}_i) > \widetilde{C}\widetilde{\beta}_g^{(p)}(\mathbf{f})$,
- $\beta_g^{(p,p')}(\mathbf{f}-\mathbf{1}_i) > \widetilde{C}\beta_g^{(p,p')}(\mathbf{f})$,
- $\beta_g^{(p,p)}(\mathbf{f}) > \widetilde{C}^{i+j}\beta_g^{(p+i,p+j)}(\mathbf{f})$.

Proof. The first two points of the lemma follow directly from Lemma 7.2.2 since
$$\widetilde{\beta}_g^{(p)}(\mathbf{f}) = (f_p+1)\beta(\mathbf{f}+\mathbf{1}_p)$$
and
$$\beta_g^{(p,p')}(\mathbf{f}) = \frac{p(f_p+1)p'(f_{p'}+\mathbb{1}_{p\neq p'})}{n+p+p'}\beta_g(\mathbf{f}+\mathbf{1}_p+\mathbf{1}_{p'}).$$

For the last point, let i such that $if_i > \delta n$. Then, by the injection that consists in gluing a particular $2i$-gon in one of the boundaries (see Figure 7.3), we have $\beta_g^{(p,p')}(\mathbf{f}) < \beta_g^{(p-1,p')}(\mathbf{f}+\mathbf{1}_i)$. Therefore, by the second point, $\beta_g^{(p,p')}(\mathbf{f}) < \frac{1}{C'}\beta_g^{(p-1,p')}(\mathbf{f})$. and we conclude by induction. □

We start by stating two "face degree transfer lemmas" (proof in Section 7.6).

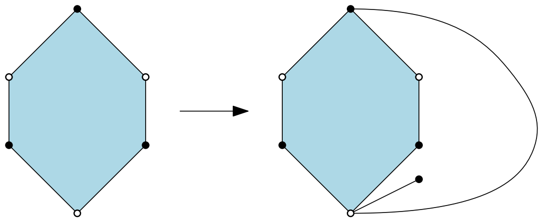


Figure 7.3 – Reducing the size of a boundary by adding a $2i$-gon (here, $i = 3$).

Lemma 7.2.5. *If $p \geqslant i$, then*

$$i f_i \beta_g(\mathbf{f} - \mathbf{1}_p) \leqslant n\beta_g(\mathbf{f} - \mathbf{1}_i)$$

Thus, if $if_i \geqslant \delta n$:

$$\beta_g(\mathbf{f} - \mathbf{1}_p) \leqslant \frac{1}{\delta}\beta_g(\mathbf{f} - \mathbf{1}_i) \tag{7.2.3}$$

Lemma 7.2.6. *If $d_1, d_2, \ldots, d_k$ are such that $1 < d_j < i$ for all $j \leqslant k$ but $\sum_j d_j \geqslant i + k - 1$, then*

$$i f_i \beta_g(\mathbf{f} - \sum_{j=1}^{k} \mathbf{1}_{d_j}) \leqslant n\beta_g(\mathbf{f} - \mathbf{1}_i)$$

Thus, if $if_i \geqslant \delta n$:

$$\beta_g(\mathbf{f} - \sum_{j=1}^{k} \mathbf{1}_{d_j}) \leqslant \frac{1}{\delta}\beta_g(\mathbf{f} - \mathbf{1}_i) \tag{7.2.4}$$

The general idea of the proof is the same as in Chapter 6, namely giving an injection that removes a small piece of a map (here, a face of degree $2i$). However, because this model is more general, there are some things to care about. First, the degrees of the faces are not bounded, so we must make sure that the operations in the injection do not involve faces of huge degrees (this is the purpose of finding "very nice edges", see below). Also, we are not guaranteed to be able to remove a face of degree exactly i, the face degree transfer lemmas ((7.2.3) and (7.2.4)) are introduced for this purpose. This is also the reason why we have two distinct bounds on faces degrees, namely A_1 and A_2. Finally, the operation itself is way more complicated and perturbates the topology of the map (contrary to Chapter 6). It is broken down in four steps for better understanding. Because of the complexity of all of this, we are not able to give a reasonable explicit formula for C.

The injection will take as input a map of $\mathcal{B}_g(\mathbf{f})$ with a marked *good set* (to be defined below, it will be proven that each map has at least $\frac{\kappa}{16}n$ good sets), and output a map of $\mathcal{B}_g(\widetilde{\mathbf{f}})$ with a marked vertex and finite information (a constant depending only on δ, κ and the function A) to go backwards, with $\widetilde{\mathbf{f}}$ either of the form $\widetilde{\mathbf{f}} = \mathbf{f} - \mathbf{1}_p$ and $i \leqslant p < A_1$ (A_1 is a constant to be defined later), or of the form $\widetilde{\mathbf{f}} = \mathbf{f} - \sum_{j=1}^{k} \mathbf{1}_{d_j}$ with $d_1, d_2, \ldots, d_k$ such that $1 < d_j < i$ for all j but $\sum d_j \geqslant i + k - 1$, and $k < A_1$. Since there are only a finite number of possibilities for $\widetilde{\mathbf{f}}$, by (7.2.3) and (7.2.4), the injection proves Lemma 7.2.2.

7.2.2 Good sets

Let $\mathbf{M} \in \mathcal{B}_g(\mathbf{f})$. It has n edges and v vertices. Take r a number that we will fix later. Let $A_1 := 2A(\min(\frac{\kappa}{32}, \delta))$. We say that an edge e of $\mathbf{M}$ is *nice* if it is not adjacent to a face of degree $> 2A_1$.

Fact. *At least* $(1 - \frac{\kappa}{16})$ *of the edges in* $\mathbf{M}$ *are nice.*

Proof. Draw an edge e of $\mathbf{M}$ uniformly at random. The face f sitting to the right of e is drawn at random with a probability proportional to its degree. Because of Property (7.2.2), f has probability $< \frac{\kappa}{32}$ to be of degree $> A_1$. The same goes with the face sitting to the left of e. □

Remark 7.2.7. *For this proof, we would only need to take* $A_1 = A(\frac{\kappa}{32})$, *but the actual definition will prove to be useful in the next section.*

Let $EG(\mathbf{M})$ be the edge-graph of $\mathbf{M}$: its vertices are the edges of $\mathbf{M}$, and two edges of $\mathbf{M}$ are connected in $EG(\mathbf{M})$ iff they share a common corner. Each edge of $\mathbf{M}$ has at most 4 corners, so $EG(\mathbf{M})$ only has vertices of degree 4 or less. In particular, let $B_r(e)$ be the ball of radius r around e in $EG(\mathbf{M})$. Then $|B_r(e)| < 4^{r+1}$. Let $A_2 := A(\frac{\kappa}{32 \cdot 4^{r+1}})$. An edge e is very nice if it is nice and no edges in $B_r(e)$ is adjacent to a face of degree $> 2A_2$. By the same argument as in the previous lemma and a union bound we have:

Fact. *At least* $(1 - \frac{\kappa}{8})$ *of the edges in* $\mathbf{M}$ *are very nice.*

Let $D = \frac{4}{\kappa}$. By (7.2.1), D is larger than twice the average degree in $\mathbf{M}$. Since at most half of the vertices have degree at least twice the average degree, and since there are $> \kappa n$ vertices, there exists a color (say white, wlog), such that:

Fact. *There are at least* $\frac{\kappa}{4}n$ *white vertices of degree* $< D$ *in* $\mathbf{M}$.

An edge is said to be *fine* if it is adjacent to a white vertex of degree $< D$ and that the face on its right is not of degree 2. By the previous fact, and since every vertex is adjacent to (at least) a face of degree > 2, there are at least $\frac{\kappa}{4}n$ fine edges in $\mathbf{M}$. An edge is said to be *good* is it is both very nice and fine. By what's above:

Lemma 7.2.8. *There are at least $\frac{\kappa}{8}n$ good edges in $\mathbf{M}$.*

We say that an A_1-uple S of edges is a *good set* if all the edges of S are good and they all are at distance $< 2r$ from each other in $EG(\mathbf{M})$.

Proposition 7.2.9. *For $r = \frac{16A_1}{\kappa} + 1$, there are at least $\frac{\kappa}{16}n$ good sets in $\mathbf{M}$.*

Proof. Let G be the set of all good edges. We can assume that for every $e \in G$, $B_r(e)$ does not contain all the edges of $\mathbf{M}$, otherwise the proposition is obviously true.

In that case, for all $e \in G$, $|B_r(e)| > r$. We are going to find a collection of distinct good sets (S_i). For this we will consider a sequence of sets of good edges (G_i), with $G_0 = G$, such that $|G_i| = |G| - i$, and $G_{i+1} \subset G_i$

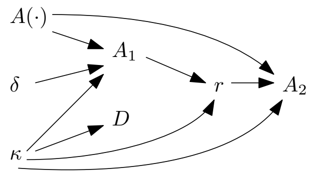
For all $i < \frac{\kappa}{16}n$, we have

$$\sum_{e \in G_i} B_r(e) > (|G| - i)r > \frac{\kappa}{16}nr > A_1 n.$$

Therefore there must be an "A_1-overlap", i.e. there exists A_1 edges whose balls of radius r all intersect (at the same point). Thus they are all at distance at most $2r$ of each other, and we just found a good set S_i. Choose $e_i \in S_i$ arbitrarily, and $G_i = G_{i-1} \setminus \{e_i\}$ (this ensures that all the S_i are distinct). And we are done. $\square$

Remark 7.2.10. *Note that the good sets are not necessarily disjoint.*

Remark 7.2.11. *It is not easy to understand the dependencies between the different constants involved. To help convince the reader there is no circularity, we provide a "causal graph" of all the variables implied.*



7.2.3 The injection

We can immediately give the proof of Lemma 7.2.2 for $i = 1$: a marked digon can be contracted into a marked edge (see Figure 7.4), and if $f_1 > \delta n$, we have

$$n\beta_g(\mathbf{f} - \mathbf{1}_i) > \delta n \beta_g(\mathbf{f})$$

which yields the result.

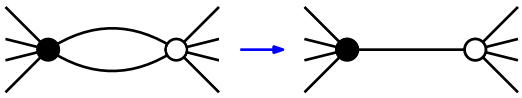


Figure 7.4 – Contraction of a digon.

Fix $i > 1$ such that $if_i > \delta n$. By the definition of A_1, $2i < A_1$. We will describe a injection that takes a map of $\mathcal{B}_g(\mathbf{f})$ and a marked good set, and returns a smaller map, still of genus g and with a marked vertex, along with some finite information describing how to reconstruct the original map.

We need to give some definitions. Take a good set S with a distinguished edge $e^* \in S$ called the *anchor*. For $e \in S \backslash \{e^*\}$, let p_e be the leftmost shortest path (in $EG(\mathbf{M})$) from e^* to e. The *path set* $P(S)$ is the union of all the p_e's. With this definition, we ensure that the union of all the edges in $P(S)$ forms a tree. We also know that $|P(S)| < 2A_1 r$, where $|P(S)|$ is the number of edges of $EG(\mathbf{M})$ in $P(S)$

Step 1: The *carving operation* of a good set S with anchor e^* consists in drawing $P(S)$ on $\mathbf{M}$ (see Figure 7.5) and deleting all the edges of $\mathbf{M}$ that are crossed by $P(S)$. Each time an edge is deleted, it merges two faces. If those faces had degree d and d', it creates a face of degree $d + d' - 1$. The "tree structure" of $P(S)$ ensures that in the end, one obtains a map $\mathbf{M}_1$ that is still connected, and has a *megaface* which is the result of the consecutive mergings of faces.

Let $i_1, i_2, \ldots, i_l$ be the degrees of the faces that were destroyed in the carving process. Then the megaface has degree $F = i_1 + i_2 + \ldots + i_l - l + 1$. We have $F < 2A_1A_2r$. In the megaface, there are less than $|P(S)|$ pairs of distinguished corners (since each pair of corners is the result of the deletion of an edge crossed by $P(S)$). The A_1 white vertices that were adjacent to good edges are all adjacent to the megaface, we will call them *good vertices*. Therefore, the information needed to reconstruct $\mathbf{M}$ from $\mathbf{M}_1$ is finite.

Step 2: The second step of the injection is the *vertex deletion*. Given a map $\mathbf{M}_1$ with a megaface and a number $d < A_1$ (d is to be fixed later

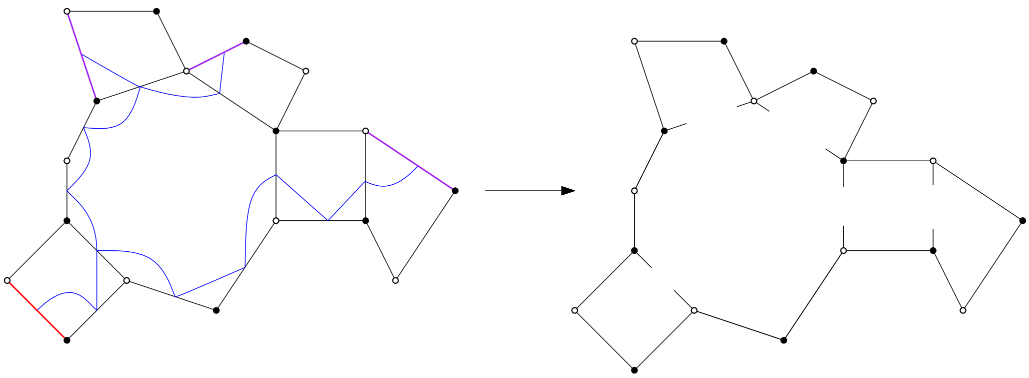


Figure 7.5 – A good set S (in in purple, anchor in red), its paths $P(S)$ in blue, and the carving operation. Marked corners are represented as dangling half edges.

and depends on $\mathbf{M}_1$), it consists in choosing d good vertices adjacent to the megaface, and deleting them as well as their adjacent edges, and marking the corresponding black corners (see Figure 7.6) for each deleted edge. Let $\mathbf{M}_2$ be the map obtained after vertex deletion. It might be disconnected, and its genus might be lower than g. $\mathbf{M}_2$ has maybe several megafaces, with the condition that in each megaface there is at least a marked corner, and that each connected component of $\mathbf{M}_2$ has at least a megaface. If $\mathbf{M}_2$ has K connected components, genus g' and m megafaces, then, by Euler's formula, we must have

$$m \leqslant (g - g') + K. \tag{7.2.5}$$

To go back to $\mathbf{M}_1$, one only needs to recreate the white vertices and reattach them to their black corners (for each vertex v there are $< D!$ possibilities for the cyclic order of edges around v, thus finite information). The vertices that were deleted had degree $< D$, therefore $\mathbf{M}_2$ has at most $A_1 D$ marked black corners. Again, only finite information is needed to go back to $\mathbf{M}_1$ from $\mathbf{M}_2$.

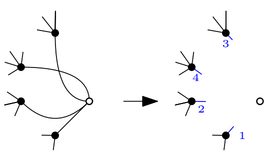


Figure 7.6 – Deleting a vertex and marking black corners

Step 3: The third step is the *reconstruction*: Take $\mathbf{M}_2$ as above, create a new white vertex v and for each mega face f of $\mathbf{M}_2$, draw an edge between v

and an arbitrary marked black corner of f (see Figure 7.7 left). One obtains a map $\mathbf{M}_3$ that is now connected and has a unique megaface, with marked black corners, and a marked white vertex inside. It has genus g'', and because of (7.2.5), we have $g'' \leqslant g$ (indeed, the genus increase from $\mathbf{M}_2$ to $\mathbf{M}_3$ is exactly $m - K$). If $g'' < g$, choose an arbitrary marked black corner and an arbitrary corner of v and attach a pair of edges as in Figure 7.7 right. Now the resulting map still has only one megaface and is of genus $g'' + 1$. Repeat if necessary to obtain a map $\mathbf{M}_4$ of genus g with a megaface and $d - 1$ less white vertices than $\mathbf{M}$. To recover $\mathbf{M}_2$ from $\mathbf{M}_4$, one has to remember which were the marked black corners in the megaface (finite information since it is a bounded number of corners in a face of bounded degree), then just delete v and its adjacent edges.

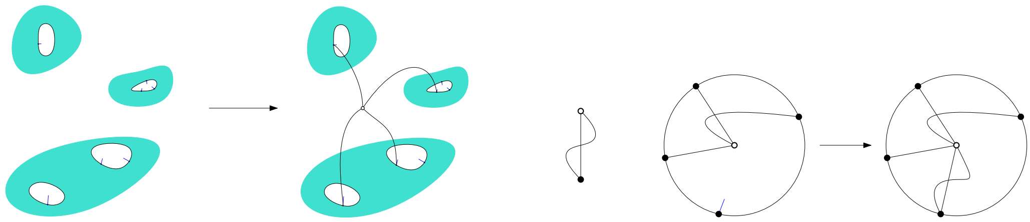


Figure 7.7 – The reconstruction. Left: reconnecting the map, right: recovering the genus.

Step 4: The final step is the *filling of the megaface*. Say the megaface of $\mathbf{M}_4$ has degree F, then the face degrees of $\mathbf{M}_4$ are given by $\mathbf{f}' + \mathbf{1}_F$, with $\mathbf{f}' \leqslant \mathbf{f}$. If the degrees of the faces missing are $i_1, i_2, \ldots, i_l$ (i.e. $\mathbf{f}' = \mathbf{f} - \mathbf{1}_{i_1} - \mathbf{1}_{i_2} - \ldots - \mathbf{1}_{i_l}$), then we must have $F = i_1 + i_2 + \ldots + i_l - (l-1) - (d-1)$.

Suppose that there exists $k < l$ such that

$$(i_1 - 1) + (i_2 - 1) + \ldots + (i_k - 1) = d - 1$$

(this is not the case in general, but in what follows we will make sure that this condition is fulfilled up to reordering the i_j's). In that case, $F - 1 = (i_{k+1} - 1) + (i_{k+2} - 2) + \ldots + (i_l - 1)$, and we can "tessellate" the megaface as in the proof of (7.2.3) (going backwards is straightforward). We end up with a map $\mathbf{M}^*$ of genus g with a marked edge, and the face degrees given by $\mathbf{f} - \sum_{j=1}^{k} \mathbf{1}_{i_j}$.

Now we can describe the injection in terms of the different operations mentioned above: take a map $\mathbf{M}$ with a marked good set S. For all $e \in S$, let d_e be the degree of the face sitting on the right of e. We have two cases:

Case 1: There exists e^* in S such that $d_{e^*} \geqslant i$. Take e^* as the anchor, and apply step 1. In the carving operation, the face sitting to the right of e^* has been destroyed. Let $d = d_{e^*}$, and apply the vertex deletion (step 2) to d good vertices, including the one that was adjacent to e^*. Reconstruct (step 3). The face degrees that are missing are $i_1, i_2, \dots, i_l$, with $i_1 = d_{e^*} = d$ (wlog). The condition for the filling of the megaface (step 4) is fulfilled. One ends up with a map $\mathbf{M}^*$ of genus g with a marked edge, and the face degrees given by $\mathbf{f} - \mathbf{1}_{d_{e^*}}$.

Case 2: $d_e < i$ for all e. Take any arbitrary ordering $e_1, e_2, \dots, e_{A_1}$ of the edges of S, and let $d_j = d_{e_j}$.

Fact. *There exists a* $1 < k < A_1$ *such that* $(i-1) < \sum_{j=1}^{k}(d_j - 1) < A_1 - 1$.

Proof. Indeed, $\sum_{j=1}^{A_1}(d_j - 1) \geqslant A_1 > i - 1$ (because $d_j \geqslant 2$ for all j). Let k be the smallest integer such that $(i-1) < \sum_{j=1}^{k}(d_j - 1)$. Then we have $\sum_{j=1}^{k-1}(d_j - 1) < i$, so since $d_k \leqslant i - 1$, we also have

$$\sum_{j=1}^{k}(d_j - 1) < 2i - 1 \leqslant A_1 - 1$$

. □

Choose the anchor arbitrarily, and apply step 1. Let $d = 1 + \sum_{j=1}^{k}(d_j - 1)$, and apply the vertex deletion (step 2) to d good vertices, including the ones that were adjacent to $e_1, e_2, \dots, e_k$. For all j, the faces adjacent to e_j is destroyed in this step. Reconstruct (step 3). The face degrees that are missing are $i_1, i_2, \dots, i_l$, with $i_j = d_j$ for all $j \leqslant k$ (wlog). The condition for the filling of the megaface (step 4) is fulfilled. One ends up with a map $\mathbf{M}^*$ of genus g with a marked edge, and the face degrees given by $\mathbf{f} - \sum_{j=1}^{k} \mathbf{1}_{d_k}$.

Lemma 7.2.2 is now proved.

7.2.4 Planarity and One-Endedness

In this section, we prove that all the potential subsequential limits are planar and one ended. The key tool of the proof (besides the Bounded Ratio Lemma), is the formula (7.1.3) proven in Chapter 5. Thanks to this formula, we have some technical estimation lemmas whose proofs will be postponed to Section 7.6.3

Fix a sequence $(g, \mathbf{f})_n$, and $\mathbf{M}_n$ uniform in $\mathcal{B}_g(\mathbf{f})$[3]. By Lemma 7.1.6, $(\mathbf{M}_n)$ is tight for d^*_{loc}. In all this section, we will denote by M a subsequential limit

[3] To make notations less cumbersome, g and $\mathbf{f}$ depend implicitly on n.

in distribution. It must be an infinite map. We will first prove that M is planar and one-ended, and then that its vertices have finite degrees. To establish planarity, the idea will be to bound, for any non-planar finite map m, the probability that $m \subset \mathbf{M}_n$ for n large. For this, we will need the following combinatorial estimate.

Lemma 7.2.12. *Fix $u \in \mathbb{N}$ and a sequence $(\mathbf{h})_n$ s.t. $\sum_i ih_i = n - u$ and $\mathbf{h} \leqslant \mathbf{f}$ for all n. Fix also $k \geqslant 2$, numbers $\ell_1, \ell_2, \ldots, \ell_k$ and perimeters p_i^j for $1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant \ell_j$. Then*

$$\sum_{\substack{\mathbf{h}^{(1)}+\mathbf{h}^{(2)}+\ldots+\mathbf{h}^{(k)}=\mathbf{h} \\ g_1+g_2+\ldots+g_k=g-1-\sum_j(\ell_j-1)}} \prod_{j=1}^{k} \beta_{g_j}^{(p_1^j,p_2^j,\ldots,p_{\ell_j}^j)}(\mathbf{h}^{(j)}) = o\left(\beta_g(\mathbf{f})\right)$$

as $n \to \infty$.

Corollary 7.2.13. *Every subsequential limit of $(\mathbf{M}_n)$ for d_{loc}^* is a.s. planar.*

Proof. If a subsequential limit M is not planar, then we can find a finite map m with holes and with genus 1 such that $m \subset M$. Indeed, if we explore M by a lazy peeling, the genus may only increase by at most 1 at each step, so if the genus is positive at some point, it must be 1 at some point. Therefore, it is enough to prove that for any such map m, we have

$$\mathbb{P}\left(m \subset \mathbf{M}_n\right) \xrightarrow[n\to+\infty]{} 0.$$

If $m \subset \mathbf{M}_n$, let $M^1, \ldots, M^k$ be the connected components of $\mathbf{M}_n \backslash m$. These components define a partition of the set of holes of m, where a hole h is in the j-th class if M^j is the connected component glued to h. Note that the number of possible partitions is finite and depends only on m (and not on n). Therefore, it is enough to prove that for any partition π of the set of holes of m, we have

$$\mathbb{P}\left(m \subset \mathbf{M}_n \text{ and the partition defined by } \mathbf{M}_n \text{ is } \pi\right) \xrightarrow[n\to+\infty]{} 0. \qquad (7.2.6)$$

If this occurs, for each j, let ℓ_j be the number of holes of M glued to M^j and let $p_1^j, \ldots, p_{\ell_j}^j$ be the perimeters of these holes. Then the connected component M^j is a map with boundary $(p_1^j, \ldots, p_{\ell_j}^j)$. Moreover, if M_j has genus g_j, then the total genus of $\mathbf{M}_n$ is equal to

$$1 + \sum_{j=1}^{k} g_j + \sum_{j=1}^{k} (\ell_j - 1),$$

so this sum must be equal to g, so

$$\sum_{j=1}^{k} g_j = g - 1 - \sum_{j=1}^{k} (\ell_j - 1).$$

Moreover, let $\mathbf{h}^{(j)}$ be the distribution of the faces in M^j, then we have $\sum_{j=1}^{k} \mathbf{h}^{(j)} = \mathbf{f} - \mathbf{s} =: \mathbf{h}$, where $\mathbf{f}$ is the distribution of the faces in M and $\mathbf{s}$ is the distribution of the faces in m.

Therefore, the number of maps $M \in \mathcal{B}_g(\mathbf{f})$ such that $m \subset M$ and the resulting partition of the holes is equal to π is the number of ways to choose, for each j, a map with boundary $(p_1^j, \dots, p_{\ell_j}^j)$, such that the total genus of these maps is $g - 1 - \sum_{j=1}^{k}(\ell_j - 1)$, and their total distribution of faces is $\mathbf{h}$. This is equal to the left-hand side of Lemma 7.2.12, so (7.2.6) is a consequence of Lemma 7.2.12, which concludes the proof. □

The proof of one-endedness will be similar, but the combinatorial estimate that is needed is slightly different.

Lemma 7.2.14.
- *Fix $u \in \mathbb{N}$ and a sequence $(\mathbf{h})_n$ s.t. $\sum_i i h_i = n - u$ and $\mathbf{h} \leqslant \mathbf{f}$ for all n. Fix also $k \geqslant 1$, numbers $\ell_1, \ell_2, \dots, \ell_k$* ***not all equal to*** *1 and perimeters p_i^j for $1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant \ell_j$. Then*

$$\sum_{\substack{\mathbf{h}^{(1)}+\mathbf{h}^{(2)}+\ldots+\mathbf{h}^{(k)}=\mathbf{h} \\ g_1+g_2+\ldots+g_k=g-1-\sum_j(\ell_j-1)}} \prod_{j=1}^{k} \beta_{g_j}^{(p_1^j, p_2^j, \dots, p_{\ell_j}^j)}(\mathbf{h}^{(j)}) = o\left(\beta_g(\mathbf{f})\right)$$

  as $n \to \infty$.

- *Fix $\mathbf{h} \leqslant \mathbf{f}$, $k \geqslant 1$, numbers $\ell_1, \ell_2, \dots, \ell_k$ and perimeters $p_1, \dots, p_k$. There is a constant C such that for every a and n we have*

$$\sum_{\substack{\mathbf{h}^{(1)}+\mathbf{h}^{(2)}+\ldots+\mathbf{h}^{(k)}=\mathbf{h} \\ g_1+g_2+\ldots+g_k=g \\ n_1, n_2 > a}} \prod_{j=1}^{k} \beta_{g_j}^{p_j}(\mathbf{h}^{(j)}) \leqslant \frac{C}{a} \beta_g(\mathbf{f})$$

  as $n \to \infty$.

Corollary 7.2.15. *Every subsequential limit of $(\mathbf{M}_n)$ for d_{loc}^* is a.s. one-ended.*

Proof. The proof is quite similar to the proof of Corollary 7.2.13, but with Lemma 7.2.14 playing the role of Lemma 7.2.12.

More precisely, if a subsequential limit M is not one-ended with positive probability, it contains a finite map m such that two of the connected components of $M\backslash m$ are infinite. This means that we can find $\varepsilon > 0$, a map m and two holes h_1, h_2 of m such that, for every $a > 0$,

$$P\left(m \subset M \text{ and } M\backslash m \text{ has two connected components with } \geqslant a \text{ edges}\right) \geqslant \varepsilon. \tag{7.2.7}$$

By Corollary 7.2.13, we can assume that m is planar. If this holds, then M contains a finite map obtained by starting from m and adding a edge in the hole h_1 and a edges in the hole h_2. We denote by $m^{a,a}$ the set of such maps. Then (7.2.7) means that for any $a > 0$, for n large enough, we have

$$\mathbb{P}\left(\mathbf{M}_n \text{ contains a map of } m^{a,a}\right) \geqslant \varepsilon. \tag{7.2.8}$$

This can occur in two different ways, which will correspond to the two items of Lemma 7.2.14:

(i) either at least one connected component of $\mathbf{M}_n\backslash m$ is adjacent to at least two holes of m,

(ii) or the k holes of m correspond to k connected components $M^1, \dots, M^k$, where M^1 and M^2 have size at least a.

In case (i), the connected components of $\mathbf{M}_n$ are maps with boundaries, at least one of which has two boundaries. The proof that the probability of this case goes to 0 is now the same as the proof of Corollary 7.2.13, but we use the first point of Lemma 7.2.14. Note that the assumption that the ℓ_j are not all 1 comes from the fact that one of the connected components is adjacent to two holes. Moreover, the sum of the genuses of the M^j is $g - \sum_j(\ell_j - 1)$ and not $g - 1 - \sum_j(\ell_j - 1)$ because this time m has genus 0 and not 1.

Similarly, in case (ii), the k holes of m must be filled with k maps with a single boundary, two of which have at least a edges, so they belong to a set of the form $\mathcal{B}_{g_j}^{p_j}(\mathbf{h}^{(j)})$ with $n_j \geqslant \frac{a}{2}$ if a is large enough compared to the perimeters of the holes. Hence, the second point of Lemma 7.2.14 allows to bound the number of ways to fill these holes. We obtain that, for a large enough, we have

$$\mathbb{P}\left(\mathbf{M}_n \text{ contains a map of } m^{a,a}\right) \leqslant o(1) + \frac{2C}{a}$$

as $n \to +\infty$, where $o(1)$ comes from case (i) and $\frac{2C}{a}$ from case (ii). This contradicts (7.2.8), so M is a.s. one-ended. $\square$

7.2.5 Finiteness of the root degree

Let M be a subsequential limit of $(\mathbf{M}_n)$ for d^*_{loc}. By Lemma 7.1.6, to finish the proof of tightness for d_{loc} (Proposition 7.2.1), we only need to show that almost surely, all the vertices of M have finite degree. As in [AS03, BL19], we will first study the degree of the root vertex, and then extend finiteness by using invariance under the simple random walk.

Lemma 7.2.16. *The root vertex of M has a.s. finite degree.*

Proof. We follow the approach of [AS03] and perform a filled-in lazy peeling exploration of M. Before specifying the peeling algorithm that we use, note that we already know by Corollary 7.2.13 that the explored part will always be planar, so no peeling step will merge two different existing holes. Moreover, by Corollary 7.2.15, if a peeling step separates the boundary into two holes, then one of them is finite and will be filled with a finite map. Therefore, at each step, the explored part will be a map with a single hole.

The peeling algorithm $\mathcal{A}$ that we use is the following: if the root vertex ρ belongs to ∂m, then $\mathcal{A}(m)$ is the edge on ∂m on the left of ρ. If $\rho \notin \partial m$, then the exploration is stopped. Since only finitely many edges incident to ρ are added at each step, it is enough to prove that the exploration will a.s. eventually stop. We recall that $\mathcal{E}_M^{\mathcal{A}}(i)$ is the explored part at time i.

We will prove that at each step, conditionally on $\mathcal{E}_M^{\mathcal{A}}(i)$, the probability to swallow the root and finish the exploration in finite time is bounded from below by a positive constant. For every map m with one hole such that $\rho \in \partial m$, we denote by m^+ the map constructed from m as follows (see Figure 7.8):

- we first glue a "face" of degree i^* (recall that i^* is the smallest $i > 1$ such that $\frac{if_i}{n} > \delta$ for n large enough) to the edge of ∂m on the left of ρ,
- we then glue the two edges of the boundary incident to ρ together,
- during the next $(i^* - 2)$ steps, at each step, pick two consecutive edges of the boundary (arbitrarily), and glue them together.

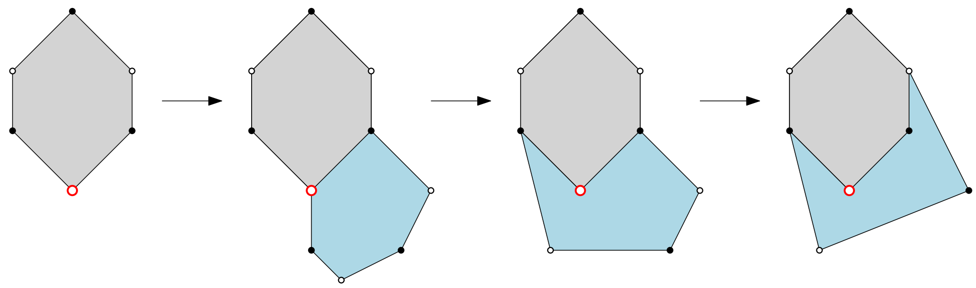


Figure 7.8 – The construction of m^+ from m. In gray, the map m. In red, the root vertex. In blue, the new face. Here, $p = i^* = 3$.

Note that m^+ is a planar map with the same perimeter as m but one more face (of degree i^*). By the choice of our peeling algorithm, if we have $\mathcal{E}_M^{\mathcal{A}}(t)^+ \subset M$, then we have $\mathcal{E}_M^{\mathcal{A}}(t+i^*) = \mathcal{E}_M^{\mathcal{A}}(t)^+$. Moreover, if this is the case, the exploration is stopped at time $t+i^*$. Hence it is enough to prove that the quantity

$$\mathbb{P}\left(m^+ \subset M | m \subset M\right)$$

is bounded from below for finite, planar maps m with a single boundary and $\rho \in \partial m$.

We fix such a m, with perimeter p, and face degrees given by $\mathbf{s}$. Let also (n_k) be a sequence of indices such that M_{n_k} converges in distribution to M. We have

$$\mathbb{P}\left(m^+ \subset M | m \subset M\right) = \lim_{k\to+\infty} \frac{\mathbb{P}\left(m^+ \in M_{n_k}\right)}{\mathbb{P}\left(m \in M_{n_k}\right)} = \lim_{k\to+\infty} \mathbb{P}(i^*, k) \frac{B_g^p\left(\mathbf{f}-\mathbf{s}-\mathbf{1}_{i^*}\right)}{B_g^p\left(\mathbf{f}-\mathbf{s}\right)},$$

where $\mathbb{P}(i^*, k)$ is the probability that a face of degree i^* is chosen in the peeling of M_{n_k}. By definition of i^*, we have

$$\mathbb{P}(i^*, k) > \delta$$

for k large enough, while the ratio $\frac{B_g^p(\mathbf{f}-\mathbf{s}-\mathbf{1}_{i^*})}{B_g^p(\mathbf{f}-\mathbf{s})}$ is bounded below uniformly by a constant because of the bounded ratio lemma, which concludes the proof.

□

Proof of Proposition 7.2.1. Let M be a subsequential limit of $(\mathbf{M}_n)$. Because of Lemma 7.1.6, it is enough to prove that almost surely, all the vertices of M have finite degrees. The argument is essentially the same as in [AS03] and relies on Lemma 7.2.16 and invariance of the distribution of $(\mathbf{M}_n)$ under the simple random walk.

□

7.3 Weakly Markovian maps

We extend Theorem 6.0.2 of Chapter 6 to general models of infinite, bipartite planar maps.

For a finite, bipartite map m with one hole, we denote by $|\partial m|$ the half-length of the hole of m. For all $j \geqslant 1$, we also denote by $v_j(m)$ the number of internal faces of m with degree $2j$.

Definition 7.3.1. Let M be a random infinite, bipartite planar map. We say that M is *weakly Markovian* if for every finite map m with one hole, the probability $\mathbb{P}(m \subset M)$ only depends on $|\partial m|$ and $(v_j(m))_{j \geqslant 1}$.

If M is weakly Markovian and $\mathbf{v} = (v_j)_{j \geqslant 1}$ is a sequence with $v_j = 0$ for j large enough, we will denote by $a^p_{\mathbf{v}}$ the probability $\mathbb{P}(m \subset M)$ for a map m with $|\partial m| = p$ and $v_j(m) = v_j$ for all j. Note that this only makes sense if there is such a map m, which is equivalent to

$$p \leqslant 1 + \sum_{j \geqslant 1} (j-1) v_j. \tag{7.3.1}$$

Therefore, if $p \geqslant 1$, we will denote by $\mathcal{V}_p$ the set of $\mathbf{v}$ satisfying (7.3.1).

In particular, by definition, the $\mathbf{q}$-IBPM is weakly Markovian, and the corresponding constants $a^p_{\mathbf{v}}$ are:

$$a^p_{\mathbf{v}}(\mathbf{q}) := C_p(\mathbf{q}) \times \mathbf{q}^{\mathbf{v}},$$

where $\mathbf{q}^{\mathbf{v}} = \prod_{j \geqslant 1} q_j^{v_j}$.

Theorem 7.3.1. *Let M be a weakly Markovian infinite bipartite planar map. Then there is a random weight sequence $\mathbf{Q} \in \mathcal{Q}_h$ such that M has the same distribution as $\mathbb{M}_{\mathbf{Q}}$. Moreover, if the degree of the root face of M has finite expectation, then $\mathbf{Q} \in \mathcal{Q}_f$ almost surely.*

We note right now that the second point of Theorem 7.3.1 is immediate once the first point is proved. Indeed, if we write $\mathrm{Rootface}(m)$ for the degree of the root face of a map m, if $\mathrm{Rootface}(M)$ has finite expectation, then

$$\mathbb{E}\left[\mathbb{E}\left[\mathrm{Rootface}(\mathbb{M}_{\mathbf{Q}}) | \mathbf{Q}\right]\right] = \mathbb{E}\left[\mathrm{Rootface}(\mathbb{M}_{\mathbf{Q}})\right] < +\infty,$$

so $\mathbb{E}\left[\mathrm{Rootface}(\mathbb{M}_{\mathbf{Q}}) | \mathbf{Q}\right] < +\infty$ a.s., so $\mathbf{Q} \in \mathcal{Q}_f$ a.s..

The first point of Theorem 7.3.1 is the natural analogue of Theorem 6.0.2 of Chapter 6, where triangulations are replaced by more general maps. The proof will rely on similar ideas, and in particular on the Hausdorff moment problem. However, two new difficulties arise:

- the random weights $\mathbf{q}$ form a family of real numbers instead of just one real number,
- in the triangular case, with the notations of Definition 7.3.1, it was obvious that all the numbers $a_{\mathbf{v}}^p$ are determined by the numbers $a_{\mathbf{v}}^1$. This is not true anymore.

The first issue can be handled by using the multi-dimensional version of the Hausdorff moment problem. The second one, on the other hand, will make the proof a bit longer than in Chapter 6. More precisely, the Hausdorff moment problem will now provide us, for every $p \geqslant 1$, a σ-finite measure μ_p on the set of weight sequences, and we will use the peeling equations to prove that all the μ_p are actually determined by μ_1.

Because of the condition (7.3.1), we will need to find a measure with suitable $\mathbf{v}$-th moments for all $\mathbf{v} \in \mathcal{V}_p$, which is slightly different than the usual Hausdorff moment problem. Therefore, we first need to state a suitable version of the moment problem, which will follow from the usual one. This is done in the next subsection.

7.3.1 The incomplete Hausdorff moment problem

We write $\mathcal{V}$ for the set of sequences $(v_j)_{j \geqslant 1}$ of natural integers such that $v_j = 0$ for j large enough. To state our version of the moment problem, we will need to consider the space of sequences u indexed by $\mathbf{v} \in \mathcal{V}_p$. For $j \geqslant 1$, we denote by Δ_j the discrete derivation operator on the j-th coordinate on this space. That is, if $u = (u_{\mathbf{v}})$, we write

$$(\Delta_j u)_{\mathbf{v}} = u_{\mathbf{v}} - u_{\mathbf{v}+\delta^j}.$$

It is easy to check that the operators Δ_j commute with each other. For all $\mathbf{k} = (k_j)_{j \geqslant 1}$ such that $k_j = 0$ for j large enough (say for $j \geqslant j_0$), we can define the operator $\Delta^{\mathbf{k}}$ by

$$\Delta^{\mathbf{k}} u = \Delta_1^{k_1} \Delta_2^{k_2} \dots \Delta_{j_0}^{k_{j_0}} u.$$

In other words, we have

$$\Delta^{\mathbf{k}} u = \sum_{\mathbf{i}} \left(\prod_{j \geqslant 1} (-1)^{i_j} \binom{k_j}{i_j} \right) u_{\mathbf{v}+\mathbf{i}},$$

where the sum is over families $\mathbf{i} = (i_j)_{j \geqslant 1}$, and the terms with a nonzero contribution are those for which $0 \leqslant i_j \leqslant k_j$ for every $j \geqslant 1$. The usual Hausdorff moment problem is then the following.

Theorem 7.3.2. *Let $(u_{\mathbf{v}})_{\mathbf{v}\in\mathcal{V}}$ be such that, for any $\mathbf{v} \in \mathcal{V}$ and any $\mathbf{k}$, we have*

$$\Delta^{\mathbf{k}} u_{\mathbf{v}} \geqslant 0.$$

Then there is a unique measure on $[0,1]^{\mathbb{N}^*}$ (equipped the product σ-algebra) such that, for all $\mathbf{v} \in \mathcal{V}$, we have

$$u_{\mathbf{v}} = \int \mathbf{q}^{\mathbf{v}} \mu(\mathrm{d}\mathbf{q}).$$

In particular μ is finite, with total mass $u_{\mathbf{0}}$.

More precisely, this is the infinite-dimensional Hausdorff moment problem, which can be deduced immediately from the finite-dimensional one by the Kolmogorov extension theorem.

For $p \geqslant 1$, we recall that $\mathcal{V}_p \subset \mathcal{V}$ is the set of $\mathbf{v} \in \mathcal{V}$ that satisfy $\sum_{j\geqslant 1}(j-1)v_j \geqslant p-1$. We also denote by $\mathcal{V}_p^*$ the set of $\mathbf{v} \in \mathcal{V}_p$ for which there is $j \geqslant 2$ such that $v_j > 0$ and $\mathbf{v} - \delta^j \in \mathcal{V}_p$. In other words $\mathcal{V}_p^*$ can be thought of as the "interior" of $\mathcal{V}_p$. Finally, we write

$$\mathcal{Q}^* = \{\mathbf{q} \in [0,1]^{\mathbb{N}^*} | \exists j \geqslant 2, q_j > 0\}.$$

Proposition 7.3.3. *We fix $p \geqslant 1$. Let $(u_{\mathbf{v}})_{\mathbf{v}\in\mathcal{V}_p}$ be a family of real numbers. We assume that for all $\mathbf{v} \in \mathcal{V}_p$ and all $\mathbf{k}$, we have*

$$\Delta^{\mathbf{k}} u_{\mathbf{v}} \geqslant 0.$$

Then there is a σ-finite measure μ on $\mathcal{Q}^*$ such that, for all $\mathbf{v} \in \mathcal{V}_p^*$, we have

$$u_{\mathbf{v}} = \int \mathbf{q}^{\mathbf{v}} \mu(\mathrm{d}\mathbf{q}).$$

Moreover, if $p = 1$, then μ is finite and $\mu(\mathcal{Q}^*) \leqslant u_{\mathbf{0}}$.

Note that this version is "weaker" than Theorem 7.3.2 in the sense that it is not always possible to have $u_{\mathbf{v}} = \int \mathbf{q}^{\mathbf{v}} \mu(\mathrm{d}\mathbf{q})$ for $\mathbf{v} \in \mathcal{V}_p \backslash \mathcal{V}_p^*$. A simple example of this phenomenon in dimension one is that the sequence $(\mathbb{1}_{i=1})_{i\geqslant 1}$ has all its discrete derivatives nonnegative. However, there is no measure on $[0,1]$ with first moment 1 and all higher moments 0. On the other hand, we obtain a stronger result on μ, since it is supported by $\mathcal{Q}^*$.

Proof. We start with the case $p = 1$. Then $\mathcal{V}_1 = \mathcal{V}$, so by Theorem 7.3.2, there is a measure $\widetilde{\mu}$ on $[0,1]^{\mathbb{N}^*}$ such that, for all $\mathbf{v} \in \mathcal{V}_1$, we have

$$u_{\mathbf{v}} = \int_{[0,1]^{\mathbb{N}^*}} \mathbf{q}^{\mathbf{v}} \widetilde{\mu}(\mathrm{d}\mathbf{q}).$$

Let μ be the restriction of $\widetilde{\mu}$ to $\mathcal{Q}^*$. If $\mathbf{v} \in \mathcal{V}_1^*$ and $\mathbf{q} \in [0,1]^{\mathbb{N}^*} \backslash \mathcal{Q}^*$, then there is $j \geqslant 2$ such that $\mathbf{v}_j > 0$ but $q_j = 0$, so $\mathbf{q}^{\mathbf{v}} = 0$ by definition. It follows that, for all $\mathbf{v} \in \mathcal{V}_1^*$, we have

$$\int_{\mathcal{Q}^*} \mathbf{q}^{\mathbf{v}} \mu(\mathrm{d}\mathbf{q}) = \int_{[0,1]^{\mathbb{N}^*}} \mathbf{q}^{\mathbf{v}} \widetilde{\mu}(\mathrm{d}\mathbf{q}) = u_{\mathbf{v}}.$$

Moreover, the total mass of μ is not larger than the total mass of $\widetilde{\mu}$, so it is at most $u_{\mathbf{0}}$.

We now assume $p \geqslant 2$. Let $\mathbf{v} \in \mathcal{V}_p$. Then $\mathbf{v}+\mathbf{w} \in \mathcal{V}_p$ for all $\mathbf{w} \geqslant 0$, so by Theorem 7.3.2, there is a finite measure $\mu_{\mathbf{v}}$ on $[0,1]^{\mathbb{N}^*}$ such that

$$u_{\mathbf{v}+\mathbf{w}} = \int \mathbf{q}^{\mathbf{w}} \mu_{\mathbf{v}}(\mathrm{d}\mathbf{q})$$

for all $\mathbf{w} \geqslant 0$. Now let $\mathbf{v}, \mathbf{v}' \in \mathcal{V}_p$. For all $\mathbf{w}$, we have

$$\int \mathbf{q}^{\mathbf{v}'} \mathbf{q}^{\mathbf{w}} \mu_{\mathbf{v}}(\mathrm{d}\mathbf{q}) = u_{\mathbf{v}+\mathbf{v}'+\mathbf{w}} = \int \mathbf{q}^{\mathbf{v}} \mathbf{q}^{\mathbf{w}} \mu_{\mathbf{v}'}(\mathrm{d}\mathbf{q}).$$

In other words, the measures $\mathbf{q}^{\mathbf{v}'} \mu_{\mathbf{v}}(\mathrm{d}\mathbf{q})$ and $q^{\mathbf{v}} \mu_{\mathbf{v}'}(\mathrm{d}\mathbf{q})$ have the same moments, so by uniqueness in Theorem 7.3.2

$$\mathbf{q}^{\mathbf{v}'} \mu_{\mathbf{v}}(\mathrm{d}\mathbf{q}) = q^{\mathbf{v}} \mu_{\mathbf{v}'}(\mathrm{d}\mathbf{q}). \tag{7.3.2}$$

In particular, for all $\mathbf{v} \in \mathcal{V}_p$, we can consider the σ-finite measure

$$\widetilde{\mu}_{\mathbf{v}}(\mathrm{d}\mathbf{q}) = \frac{\mu_{\mathbf{v}}(\mathrm{d}\mathbf{q})}{\mathbf{q}^{\mathbf{v}}}$$

defined on $\{\mathbf{q}^{\mathbf{v}} > 0\}$. Then (7.3.2) implies that, for any $\mathbf{v}$ and $\mathbf{v}'$, the measures $\widetilde{\mu}_{\mathbf{v}}$ and $\widetilde{\mu}_{\mathbf{v}'}$ coincide on $\{\mathbf{q}^{\mathbf{v}} > 0\} \cap \{\mathbf{q}^{\mathbf{v}'} > 0\}$. Therefore, there is a measure μ on $\bigcup_{\mathbf{v} \in \mathcal{V}_p} \{\mathbf{q}^{\mathbf{v}} > 0\} = \mathcal{Q}^*$ such that, for all $\mathbf{v} \in \mathcal{V}_p$, we have

$$\mu_{\mathbf{v}}(\mathrm{d}\mathbf{q}) = \mathbf{q}^{\mathbf{v}} \mu(\mathrm{d}\mathbf{q}) \quad \text{ on } \{\mathbf{q}^{\mathbf{v}} > 0\}. \tag{7.3.3}$$

Since μ is finite on $\{q_j > \varepsilon\}$ for all $\varepsilon > 0$ and $j \geqslant 2$, it is σ-finite. We would now like to extend (7.3.3) to all $\mathcal{Q}^*$ for $\mathbf{v} \in \mathcal{V}_p^*$.

For this, let $\mathbf{v} \in \mathcal{V}_p^*$, and let $j \geqslant 2$ be such that $v_j > 0$ and $\mathbf{v} - \delta^j \in \mathcal{V}_p$. We have $p\delta^j \in \mathcal{V}_p$, so we can apply (7.3.2) to $\mathbf{v}$ and $p\delta^j$. We obtain, on $\{q_j > 0\}$:

$$\mu_{\mathbf{v}}(\mathrm{d}\mathbf{q}) = \mathbf{q}^{\mathbf{v}} \frac{\mu_{p\delta^j}(\mathrm{d}\mathbf{q})}{q_j^p} = \mathbf{q}^{\mathbf{v}} \mu(\mathrm{d}\mathbf{q}),$$

using also (7.3.3) for $p\delta^j$. In other words, (7.3.3) holds on $\{q_j > 0\}$.

On the other hand, for $\mathbf{v}$ and $\mathbf{v} - \delta^p$, we can obtain a stronger version of (7.3.2). More precisely, for all $\mathbf{w}$, we have

$$\int q_j \mathbf{q}^{\mathbf{w}} \mu_{\mathbf{v}-\delta^j}(\mathrm{d}\mathbf{q}) = u_{\mathbf{v}+\mathbf{w}} = \int \mathbf{q}^{\mathbf{w}} \mu_{\mathbf{v}}(\mathrm{d}\mathbf{q}),$$

so the measures $q_j \mu_{\mathbf{v}-\delta^j}(\mathrm{d}\mathbf{q})$ and $\mu_{\mathbf{v}}(\mathrm{d}\mathbf{q})$ have the same moments, so they coincide. But the first one is 0 on $\{q_j = 0\}$, so it is also the case for the second. Therefore, (7.3.3) holds on $\{q_j = 0\}$, with both sides equal to 0.

Finally, we have proved that (7.3.3) holds on $[0,1]^{\mathbb{N}^*}$. By integrating over $[0,1]^{\mathbb{N}^*}$ and using that the total mass of $\mu_{\mathbf{v}}$ is $u_{\mathbf{v}}$, we get the result. $\square$

7.3.2 Proof of Theorem 7.3.1

As in Chapter 6, we start by writing down the peeling equations linking the numbers $a^p_{\mathbf{v}}$ together. For every $p \geqslant 1$ and $\mathbf{v} \in \mathcal{V}_p$, we have

$$a^p_{\mathbf{v}} = \sum_{j \geqslant 1} a^{p+j-1}_{\mathbf{v}+\delta^j} + 2 \sum_{i=1}^{p-1} \sum_{\mathbf{w} \in \mathcal{V}} \# \mathcal{M}_{i-1,\mathbf{w}} a^{p-i}_{\mathbf{v}+\mathbf{w}}, \tag{7.3.4}$$

where $\mathcal{M}_{p,\mathbf{w}}$ is the set of bipartite maps of the $2p$-gon with exactly w_j internal faces of degree $2j$ for all $j \geqslant 1$. These equations, together with the facts that $a^1_{\mathbf{0}} = 1$ and $a^p_{\mathbf{v}} \geqslant 0$, characterize the families $(a^p_{\mathbf{v}})$ of numbers that may arise from a weakly Markovian map. In order to be able to use the Hausdorff moment problem like in Chapter 6, we now need to check that the discrete derivatives of $(a^p_{\mathbf{v}})$ are nonnegative.

Lemma 7.3.4. *Let M be a weakly Markovian bipartite map, and let $a^p_{\mathbf{v}}$ be the associated constants. For every $\mathbf{k} \geqslant 0$, $p \geqslant 1$ and $\mathbf{v} \in \mathcal{V}_p$, we have*

$$\left(\Delta^{\mathbf{k}} a^p\right)_{\mathbf{v}} \geqslant 0.$$

Proof. The proof is very similar to the proof of Lemma 16 in Chapter 6, with the following modification: in Chapter 6, it was useful that in the same peeling equation, we had a^p_v appearing on the left and a^p_{v+1} on the right. However, in (7.3.4) $a^p_{\mathbf{v}+\delta^j}$ does not appear in the right-hand side (this is because we are using the lazy peeling process of [Bud15] instead of the simple peeling of [Ang03]). Therefore, instead of using directly the peeling equation, we will need to use the *double peeling equation*, which corresponds

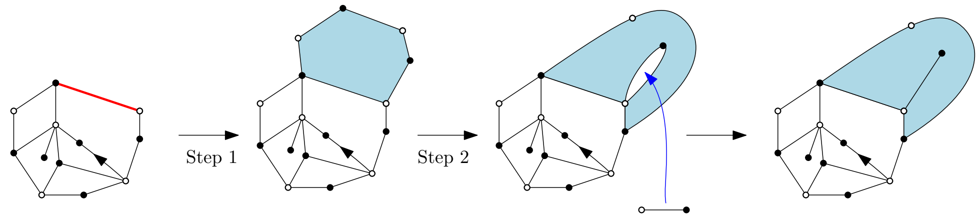


Figure 7.9 – In two peeling steps, the perimeter stays constant and one face with degree $2j$ is added (here $j = 3$).

to performing two peeling steps (instead of one in (7.3.4), because now it is lazy peeling).

More precisely, the peeling equation (7.3.4) gives an expansion of $a^p_{\mathbf{v}}$. The *double peeling equation* is obtained from (7.3.4) by replacing all the terms in the right-hand side by their expansion given by (7.3.4). Note that this indeed makes sense because if $(p, \mathbf{v})$ satisfies 7.3.1, then $(p + j - 1, \mathbf{v} + \delta^j)$ satisfies (7.3.1) as well for all $j \geqslant 1$, and so does $(p - i, \mathbf{v} + \mathbf{w})$ for $i \geqslant 1$ and $\mathbf{w} \in \mathcal{V}$.

The equation we obtain is of the form

$$a^p_{\mathbf{v}} = \sum_{i \in \mathbb{Z},\, \mathbf{w} \in \mathcal{V}} c^{p,i}_{\mathbf{v},\mathbf{w}} a^{p+i}_{\mathbf{v}+\mathbf{w}}, \tag{7.3.5}$$

where the coefficients $c^{p,i}_{\mathbf{v},\mathbf{w}}$ are nonnegative integers. An explicit formula for these could easily be computed in terms of the $|\mathcal{M}_{i,\mathbf{w}}|$, but this will not be needed. Here are the facts that will be useful:

(i) the coefficients $c^{p,i}_{\mathbf{v},\mathbf{w}}$ actually do not depend on $\mathbf{v}$, so we can write them $c^{p,i}_{\mathbf{w}}$,

(ii) we have $c^{p,2j-2}_{2\delta^j} = 1$ for every $j \geqslant 1$,

(iii) we have $c^{p,0}_{\delta^j} \geqslant 1$ for every $j \geqslant 2$.

The second item expresses the fact that, for a given peeling algorithm, there is a unique way to obtain a map with perimeter $p + 2j - 2$ with faces $\mathbf{v} + 2\delta^j$ in two peeling steps. This way is to discover a unique face of degree $2j$ at both steps. The third item means that it is possible to obtain in two peeling steps a map with the same perimeter but one more face of degree $2j$. For this, we can discover a new face of degree $2j$ at the first step, and glue all but two sides of this face two by two at the second step (see Figure 7.9).

We now prove the lemma by induction on $|\mathbf{k}| = \sum_{j\geqslant 1} k_j$. First, the case $|\mathbf{k}| = 0$ just means that $a^p_{\mathbf{v}} \geqslant 0$ for all $(p, \mathbf{v})$ satisfying (7.3.1), which is immediate. Let us now assume that the lemma is true for $\mathbf{k}$ and prove it for $\mathbf{k} + \delta^j$, where $j \geqslant 1$. We will first treat the case where $j \geqslant 2$. Using the double peeling equation (7.3.5) for $(p, \mathbf{v} + \mathbf{i})$ for different values of $\mathbf{i}$, we have

$$\left(\Delta^{\mathbf{k}} a^p\right)_{\mathbf{v}} = \sum_{i\in\mathbb{Z},\, \mathbf{w} in \mathcal{V}} c^{p,i}_{\mathbf{w}} \left(\Delta^{\mathbf{k}} a^{p+i}\right)_{\mathbf{v}+\mathbf{w}}.$$

Therefore, using the induction hypothesis and Item (ii) above, we can write

$$\begin{aligned}
0 &\leqslant \left(\Delta^{\mathbf{k}} a^{p+2j-2}\right)_{\mathbf{v}+2\delta^j} \\
&= c^{p,2j-2}_{2\delta^j} \left(\Delta^{\mathbf{k}} a^{p+2j-2}\right)_{\mathbf{v}+2\delta^j} \\
&= \left(\Delta^{\mathbf{k}} a^p\right)_{\mathbf{v}} - \sum_{(i,\mathbf{w})\neq(2j-2,2\delta^j)} c^{p,i}_{\mathbf{w}} \left(\Delta^{\mathbf{k}} a^{p+i}\right)_{\mathbf{v}+\mathbf{w}}.
\end{aligned}$$

Using the induction hypothesis again, we can remove all the terms in the last sum except the one where $(i, \mathbf{w}) = (0, \delta^j)$. Using Item (iii) above, we obtain

$$0 \leqslant \left(\Delta^{\mathbf{k}} a^p\right)_{\mathbf{v}} - \left(\Delta^{\mathbf{k}} a^p\right)_{\mathbf{v}+\delta^j} = \left(\Delta^{\mathbf{k}+\delta^j} a^p\right)_{\mathbf{v}},$$

which proves the induction step for $j \geqslant 2$. If $j = 1$, Item (iii) is not true anymore (it is not possible to add only one face of degree 2 in 2 steps without changing the perimeter). Therefore, instead of (7.3.5), we use the simple peeling equation (7.3.4) like in Chapter 6. More precisely, in the induction step, we fix $j' \geqslant 2$ and write, using (7.3.4):

$$\begin{aligned}
0 &\leqslant \left(\Delta^{\mathbf{k}} a^{p+j'-1}\right)_{\mathbf{v}+\delta_{j'}} \\
&= \left(\Delta^{\mathbf{k}} a^p\right)_{\mathbf{v}} - \sum_{j''\neq j'} \left(\Delta^{\mathbf{k}} a^{p+j''-1}\right)_{\mathbf{v}+\delta_{j''}} - 2\sum_{i=0}^{p-1} \sum_{\mathbf{w}} |\mathcal{M}_{i-1,\mathbf{w}}| \left(\Delta^{\mathbf{k}} a^{p-i}\right)_{\mathbf{v}+\mathbf{w}}.
\end{aligned}$$

each term in the two sums is nonnegative by the induction hypothesis, so we can remove the second sum and keep only the term $j'' = 1$ in the first one to obtain

$$0 \leqslant \left(\Delta^{\mathbf{k}} a^p\right)_{\mathbf{v}} - \left(\Delta^{\mathbf{k}} a^p\right)_{\mathbf{v}+\delta_1} = \left(\Delta^{\mathbf{k}+\delta_1} a^p\right)_{\mathbf{v}}.$$

This concludes the proof of the lemma. □

By Lemma 7.3.4 and Proposition 7.3.3, for all $p \geqslant 1$, there is a σ-finite measure μ_p on $\mathcal{Q}^*$ such that, for all $\mathbf{v} \in \mathcal{V}^*_p$,

$$a^p_{\mathbf{v}} = \int_{\mathcal{Q}^*} \mathbf{q}^{\mathbf{v}} \mu_p(\mathrm{d}\mathbf{q}) \tag{7.3.6}$$

and furthermore $\mu_1(\mathcal{Q}^*) \leqslant a^1_{\mathbf{0}} = 1$. We now replace $a^p_{\mathbf{v}}$ by this expression in the peeling equation (7.3.4). It is easy to check that, if $\mathbf{v} \in \mathcal{V}^*_p$, then for all the terms $a^{p'}_{\mathbf{v}'}$ appearing in the right-hand side of the peeling equation for $(p, \mathbf{v})$, we also have $\mathbf{v}' \in \mathcal{V}^*_{p'}$, so we can replace all the terms of the peeling equation using (7.3.6). We obtain

$$\begin{aligned}\int \mathbf{q}^{\mathbf{v}}\, \mu_p(\mathrm{d}\mathbf{q}) &= \sum_{j \geqslant 1} \int \mathbf{q}^{\mathbf{v}+\delta^j}\, \mu_{p+j-1}(\mathrm{d}\mathbf{q}) + 2\sum_{i=1}^{p-1} \sum_{\mathbf{w} \geqslant 0} |\mathcal{M}_{i-1,\mathbf{w}}| \int \mathbf{q}^{\mathbf{v}+\mathbf{w}}\, \mu_{p-i}(\mathrm{d}\mathbf{q}) \\ &= \int \mathbf{q}^{\mathbf{v}} \left(\sum_{j \geqslant 1} q_j\, \mu_{p+j-1}(\mathrm{d}\mathbf{q}) + 2\sum_{i=1}^{p-1} W_{i-1}(\mathbf{q})\, \mu_{p-i}(\mathrm{d}\mathbf{q}) \right),\end{aligned}$$

where we recall that $W_{i-1}(\mathbf{q})$ is the partition function of Boltzmann maps of the $2(i-1)$-gon with Boltzmann weights $\mathbf{q}$. In particular, the right-hand side for $i = 2$ must be finite, which means that $\mu_p(\mathcal{Q}^* \backslash \mathcal{Q}_a) = 0$, where $\mathcal{Q}_a$ is the set of admissible weight sequences. In other words, μ_p is actually a measure on the set of admissible weight sequences. Moreover, the last display means that the two measures

$$\mu_p(\mathrm{d}\mathbf{q}) \text{ and } \nu_p(\mathrm{d}\mathbf{q}) = \sum_{j \geqslant 1} q_j\, \mu_{p+j-1}(\mathrm{d}\mathbf{q}) + 2\sum_{i=1}^{p-1} W_{i-1}(\mathbf{q})\, \mu_{p-i}(\mathrm{d}\mathbf{q})$$

have the same $\mathbf{v}$-th moment for all $\mathbf{v} \in \mathcal{V}^*_p$. In particular, if we fix $j \geqslant 2$, this is true as soon as $v_j \geqslant p$, so the measures $q_j^p \mu_p(\mathrm{d}\mathbf{q})$ and $q_j^p \nu_p(\mathrm{d}\mathbf{q})$ have the same moments so they are equal, so μ_p and ν_p coincide on $\{q_j > 0\}$. Since this is true for all $j \geqslant 2$ and μ_p, ν_p are defined on $\mathcal{Q}^* = \bigcup_{j \geqslant 2} \{q_j > 0\}$, the measures μ_p and ν_p are the same, that is,

$$\mu_p(\mathrm{d}\mathbf{q}) = \sum_{j \geqslant 1} q_j\, \mu_{p+j-1}(\mathrm{d}\mathbf{q}) + 2\sum_{i=1}^{p-1} W_{i-1}(\mathbf{q})\, \mu_{p-i}(\mathrm{d}\mathbf{q}). \tag{7.3.7}$$

We now note that this equation is very similar to the one satisfied by the constants $C_p(\mathbf{q})$ used to define the $\mathbf{q}$-IBPM. More precisely, we fix a measure μ such that all the μ_p are absolutely continuous with respect to μ (take e.g. $\mu(\mathrm{d}\mathbf{q}) = \sum_{p \geqslant 1} \frac{g_p(\mathbf{q})\mu_p(\mathrm{d}\mathbf{q})}{2^p}$, where $g_p(\mathbf{q}) > 0$ is such that the total mass of $g_p(\mathbf{q})\mu_p(\mathrm{d}\mathbf{q})$ is at most 1). We denote by $f_p(\mathbf{q})$ the density of μ_p with respect to μ. Then (7.3.7) becomes

$$f_p(\mathbf{q}) = \sum_{j \geqslant 1} q_j f_{p+j-1}(\mathbf{q}) + 2\sum_{i=1}^{p-1} W_{i-1}(\mathbf{q}) f_{p-i}(\mathbf{q})$$

for μ-almost every $\mathbf{q} \in \mathcal{Q}^*$. In other words, $\left(f_p(\mathbf{q})\right)_{p \geqslant 1}$ satisfies the exact same equation as $\left(C_p(\mathbf{q})\right)_{p \geqslant 1}$ in [Bud18a, Appendix C]. These equations have a nonzero solution if and only if $\mathbf{q} \in \mathcal{Q}_h$, so the measures μ_p are actually on $\mathcal{Q}_h$. Moreover, by uniqueness of the solution (up to a multiplicative constant), we have

$$f_p(\mathbf{q}) = \frac{C_p(\mathbf{q})}{C_1(\mathbf{q})} f_1(\mathbf{q}) = C_p(\mathbf{q}) f_1(\mathbf{q})$$

for μ-almost every $\mathbf{q}$, so $\mu_p(\mathrm{d}\mathbf{q}) = C_p(\mathbf{q})\mu_1(\mathrm{d}\mathbf{q})$. Now let $\alpha \leqslant 1$ be the total mass of the measure μ_1, and let $\mathbf{Q}$ be a random variable with distribution $\alpha^{-1}\mu$. We then have, for all $p \geqslant 1$ and $\mathbf{v} \in \mathcal{V}_p^*$, is m is a map with perimeter p and face degrees $\mathbf{v}$:

$$\mathbb{P}(m \subset M) = a_{\mathbf{v}}^p = \int \mathbf{q}^{\mathbf{v}} \mu_p(\mathrm{d}\mathbf{q}) = \alpha \mathbb{E}\left[C_p(\mathbf{Q})\mathbf{Q}^{\mathbf{v}}\right] = \alpha \mathbb{P}\left(m \subset \mathbb{M}_{\mathbf{Q}}\right). \quad (7.3.8)$$

Note that $\mathbf{Q}$ is not well-defined if $\alpha = 0$, but in this case $\mu_p = 0$ for all p so (7.3.8) remains true for any choice of $\mathbf{Q}$. To conclude that M has the law of $\mathbb{M}_{\mathbf{Q}}$, all we have left to prove is that $\alpha = 1$ and that (7.3.8) can be extended to any $\mathbf{v} \in \mathcal{V}_p$. For this, we will use the fact that, when we explore M via a peeling exploration, the perimeter and volumes of the explored region at time n satisfy $\mathbf{v} \in \mathcal{V}_p^*$ for n large enough.

More precisely, if $\mathcal{A}$ is a peeling algorithm, we denote by $\mathcal{E}_n^{\mathcal{A}}(M)$ the explored part of M after n steps of a peeling exploration according to $\mathcal{A}$. We denote by P_n the half-perimeter of the hole of $\mathcal{E}_n^{\mathcal{A}}(M)$ and by $\mathbf{V}_n$ the sequence of degrees of its internal faces (that is, $V_{n,j}$ is the number of internal faces of $\mathcal{E}_n^{\mathcal{A}}(M)$ with degree $2j$). Since M is weakly Markovian, the process $(P_n, \mathbf{V}_n)_{n \geqslant 0}$ is a Markov chain whose law does not depend on the peeling algorithm $\mathcal{A}$.

Lemma 7.3.5. *We have*

$$\mathbb{P}\left(\mathbf{V}_n \in \mathcal{V}_{P_n}^*\right) \xrightarrow[n \to +\infty]{} 1.$$

Proof. Since the probability in the lemma does not depend on $\mathcal{A}$, it is sufficient to prove the result for a particular peeling algorithm. Therefore, we can assume that $\mathcal{A}$ has the following property: if the root face of m and its hole have a common vertex m, then the peeled edge $\mathcal{A}(m)$ is incident to such a vertex. We will prove that for this algorithm, we have a.s. $\mathbf{V}_n \in \mathcal{V}_{P_n}^*$ for n large enough.

More precisely, since the vertex degrees of M are a.s. finite and by definition of $\mathcal{A}$, all the vertices incident to the root face will eventually disappear from the boundary of the explored part. Therefore, for n large enough, no vertex incident to the root face is on $\partial\mathcal{E}_n^{\mathcal{A}}(M)$. We now fix n with this property. If we denote by $\mathrm{Inn}(m)$ the number of internal vertices of a map with a hole m and by $2J$ the degree of the root face of M, this implies $\mathrm{Inn}\left(\mathcal{E}_n^{\mathcal{A}}(M)\right) \geqslant 2J$ for n large enough.

On the other hand, the total number of edges of $\mathcal{E}_n^{\mathcal{A}}(M)$ is $p+\sum_{j\geqslant 1} jV_{n,j}$, so by the Euler formula, we have

$$\begin{aligned}\mathrm{Inn}\left(\mathcal{E}_n^{\mathcal{A}}(M)\right) &= 2 + \left(P_n + \sum_{j\geqslant 1} jV_{n,j}\right) - \left(\sum_{j\geqslant 1} V_{n,j} + 1\right) - 2P_n\\ &= 1 - P_n + \sum_{j\geqslant 1}(j-1)V_{n,j}.\end{aligned}$$

Taking n large enough to have $\mathrm{Inn}\left(\mathcal{E}_n^{\mathcal{A}}(M)\right) \geqslant 2J$, we obtain

$$\left(\sum_{j\geqslant 1}(j-1)V_{n,j}\right) - (J-1) \geqslant (2J+P_n-1) - (J-1) = P_n + J > P_n - 1,$$

so $V_{n,J} > 0$ and $\mathbf{V} - \delta^J \in \mathcal{V}_{P_n}$. This proves $V_{n,J} \in \mathcal{V}_{P_n}^*$ for n large enough. □

We now conclude the proof of Theorem 7.3.1 from (7.3.8). We consider a finite map with a hole m_0 and a peeling algorithm $\mathcal{A}$ that is consistent with m_0 in the sense that m_0 is a possible value of $\mathcal{E}_{n_0}^{\mathcal{A}}$ for some $n_0 \geqslant 0$. We note that $\mathcal{E}_{n_0}^{\mathcal{A}}(M) = m_0$ if and only if $m_0 \subset M$. Indeed, the direct implication is immediate. The indirect one comes from the fact that, if $m_0 \subset M$, then all the peeling steps until time n_0 must be consistent with m_0, so $m_0 \subset M$ determines the first n_0 peeling steps. We now take $n \geqslant n_0$. We sum (7.3.8) over all possible values m of $\mathcal{E}_n^{\mathcal{A}}(M)$ such that $m_0 \subset m$ and the perimeter p and internal face degrees $\mathbf{v}$ of m satisfy $\mathbf{v} \in \mathcal{V}_p^*$. We get

$$\mathbb{P}\left(m_0 \subset M \text{ and } \mathbf{V}_n \in \mathcal{V}_{P_n}^*\right) = \alpha\mathbb{P}\left(m_0 \subset \mathbb{M}_{\mathbf{Q}} \text{ and } \mathbf{V}_n^{\mathbf{Q}} \in \mathcal{V}_{P_n^{\mathbf{Q}}}^*\right),$$

where $P_n^{\mathbf{Q}}$ and $\mathbf{V}_n^{\mathbf{Q}}$ are the analogues of P_n and $\mathbf{V}_n$ for $\mathbb{M}_{\mathbf{Q}}$ instead of M. Since $\mathbb{M}_{\mathbf{Q}}$ is weakly Markovian, we can apply Lemma 7.3.5 to both M and $\mathbb{M}_{\mathbf{Q}}$. Therefore, letting $n \to +\infty$ in the last display, we get

$$\mathbb{P}\left(m_0 \subset M\right) = \alpha\mathbb{P}\left(m_0 \subset \mathbb{M}_{\mathbf{Q}}\right)$$

for all m_0. In particular, if m_0 is the trivial map consisting only of the root edge, we get $\alpha = 1$, so M and $\mathbb{M}_{\mathbf{Q}}$ have the same law.

7.4 The parameters are deterministic

7.4.1 The two holes argument

Recall the context: we fix $(\alpha_i)_{i\geqslant 1}$ satisfying $\sum_i i\alpha_i = 1$ and $\sum i^2\alpha_i < +\infty$, and $0 \leqslant \theta < \frac{1}{2}(1-\sum_i \alpha_i)$. We also fix (g_n) with $\frac{g_n}{n} \to \theta$ and numbers f_i^n such that

$$\sum_i i f_i^n = n \text{ and } \frac{f_i^n}{2n} \xrightarrow[n\to+\infty]{} \alpha_i. \tag{7.4.1}$$

We denote by $\mathbf{M}_n$ a uniform variable on $\mathcal{B}_{g_n}(\mathbf{f}^{(n)})$. By the results of the previous sections, we have:

Theorem 7.4.1. *The sequence $(\mathbf{M}_n)$ is tight for the local topology. Moreover, any subsequential limit of $(\mathbf{M}_n)$ is a mixture of infinite Boltzmann planar maps, i.e. it is of the form $\mathbb{M}_{\mathbf{Q}}$, where $\mathbf{Q}$ is a random variable with values in $\mathcal{Q}_{\{}$.*

Remark 7.4.2. *The previous theorem also holds if we remove the condition $\sum i^2\alpha_i < +\infty$. In this case, $\mathbf{Q}$ has values in $\mathcal{Q}_h$.*

Let $(\mathbf{M}_n, e_n^1, e_n^2)$ be a uniform, birooted map (i.e. e_n^1 and e_n^2 are picked uniformly and independently among the edges of $\mathbf{M}_n$). We will consider that n lies in a subsequence along which $(\mathbf{M}_n, e_n^2)$ converges in distribution to $\mathbb{M}^1_{\mathbf{Q}^1}$. Then up to further extraction, we can assume the joint convergence

$$\Big((\mathbf{M}_n, e_n^1), (\mathbf{M}_n, e_n^2)\Big) \xrightarrow[n\to+\infty]{(d)} \Big(\mathbb{M}^1_{\mathbf{Q}^1}, \mathbb{M}^2_{\mathbf{Q}^2}\Big)$$

for the local topology, where $\mathbf{Q}^1$ and $\mathbf{Q}^2$ have the same distribution. Moreover, by the Skorokhod representation theorem, we can assume this joint convergence is almost sure. The goal of this section is to prove the next result.

Theorem 7.4.3. *We have $\mathbf{Q}^1 = \mathbf{Q}^2$ almost surely.*

Corollary 7.4.4. *Under the assumptions (7.4.1), let $\mathbb{M}_{\mathbf{Q}}$ be a subsequential limit of $(\mathbf{M}_n)$. Then almost surely, we have*

$$d(\mathbf{Q}) = \frac{1}{2}\left(1 - 2\theta - \sum_i \alpha_i\right) \text{ and, for all } i \geqslant 1,\ a_i(\mathbf{Q}) = \alpha_i.$$

7.4.2 Finding two pieces with the same perimeter

We now fix a deterministic peeling algorithm $\mathscr{A}$, and let $p, t_1 > 0$. Given $(\mathbf{M}_n, e_n^1, e_n^2)$, we:

- explore the map m around e_n^1 according to $\mathscr{A}$ until either we have made t_1 peeling steps, or the perimeter of the explored region is exactly equal to p, and denote by τ_n^1 the time at which we stop;
- do the same thing around e^2 (and denote by τ_n^2 the stopping time).

We write $\mathcal{E}_{n,p,v_0}$ for the event where both τ_n^1 and τ_n^2 are smaller than v_0, and where the two regions explored around e_n^1 and e_n^2 are disjoint.

Our first goal will be to prove the next result. It roughly means that, provided t_1 is large enough, the probability of $\mathcal{E}_{n,p,t_1}$ is not too small, even if we condition on $\mathbf{Q}^1$ and $\mathbf{Q}^2$. More precisely, we recall that the functions r_j of $\mathbf{q}$ for $j \in \mathbb{N}^* \cup \{\infty\}$ are defined in Proposition 7.1.2.

Proposition 7.4.5. *We fix $j \in \mathbb{N}^* \cup \{\infty\}$, and $\varepsilon > 0$. Then there is $\delta > 0$ with the following property. For every $p > 0$ large enough, there is v_0 such that, for n large enough:*

$$\textit{if } \mathbb{P}\left(|r_j(\mathbf{Q}^1) - r_j(\mathbf{Q}^2)| > \varepsilon\right) \geqslant \varepsilon,$$

$$\textit{then } \mathbb{P}\left(|r_j(\mathbf{Q}^1) - r_j(\mathbf{Q}^2)| > \frac{\varepsilon}{2} \textit{ and } (\mathbf{M}_n, e_n^1, e_n^2) \in \mathcal{E}_{n,p,t_1}\right) \geqslant \delta.$$

Note that δ depends on the numbers α_i. Note also that we used the Skorokhod theorem to couple the finite and infinite maps together, so the last event makes sense.

Here is why Proposition 7.4.5 seems reasonable: conditionally on $(\mathbf{Q}^1, \mathbf{Q}^2)$, the perimeters of the explored region along a peeling exploration of $\mathbb{M}^1_{\mathbf{Q}^1}$ and $\mathbb{M}^2_{\mathbf{Q}^2}$ are random walks conditioned to stay positive. Moreover, since $\mathbf{Q} \in \mathcal{Q}_f$, these random walks do not have a too heavy tail, so they have a reasonable choice of hitting exactly p. However, an important issue is that there is no reason a priori why $\mathbb{M}^1$ and $\mathbb{M}^2$ should be independent conditionally on $\mathbf{Q}^1$ and $\mathbf{Q}^2$. Therefore, the sketch of the proof will be the following:

- we fix a large constant $C > 0$ (keep in mind $C << p$),
- we prove that both walks have a large probability to hit the interval $[p, p + C]$ in time $t_0(p)$ (this is roughly the content of Lemma 7.4.6),

- once both explorations around e_n^1 and e_n^2 in $\mathbf{M}_n$ have hit $[p, p+C]$, we use the bounded ratio lemma to show that, with probability bounded from below by roughly e^{-C}, both perimeters fall to exactly p in at most C steps. This will prove the proposition with $\delta \approx \varepsilon e^{-C}$ and $t_1 = t_0(p) + C$.

The point of replacing p by $[p, p+C]$ is to deal with events of large probability, so that we don't need any independence to make sure two events simultaneously happen.

For this, consider our exploration around e_n^1 according to $\mathcal{A}$. We denote by $\sigma^{1,n}_{[p,p+C]}$ the first time at which the perimeter is in $[p, p+C]$ (this stopping time might be infinite). We define $\sigma^{1,n}_{[p,p+C]}$ (resp. $\sigma^{1,\infty}_{[p,p+C]}$, $\sigma^{2,\infty}_{[p,p+C]}$) as the analogue quantity for the exploration in $\mathbf{M}_n$ around e_n^2 (resp. in $\mathbb{M}^1_{\mathbf{Q}^1}$, in $\mathbb{M}^2_{\mathbf{Q}^2}$).

Lemma 7.4.6. *We have*

$$\lim_{C \to +\infty} \liminf_{p \to +\infty} \mathbb{P}\left(\sigma^{1,\infty}_{[p,p+C]} < +\infty\right) = 1.$$

Proof. Note that we know that $\mathbf{Q}^1$ almost surely satisfies $\sum_i i^{3/2} \mathbf{Q}^1_i < +\infty$, so it is enough to prove that, for any weight sequence $\mathbf{q} = (q_i)$ satisfying $\sum_i i^{3/2} q_i < +\infty$, we have

$$\lim_{C \to +\infty} \liminf_{p \to +\infty} \mathbb{P}\left(\sigma^{1,\infty}_{[p,p+C]} < +\infty \middle| \mathbf{Q} = \mathbf{q}\right) = 1. \tag{7.4.2}$$

The lemma then follows by taking the expectation and using Fatou's lemma. Note that conditionally on $\mathbf{Q}^1 = \mathbf{q}$, the law of $\mathbb{M}^1_{\mathbf{Q}^1}$ is the law of $\mathbb{M}_{\mathbf{q}}$. In particular, the process P describing the perimeter of the explored region is a random walk X conditioned to stay positive.

To prove (7.4.2), we distinguish two cases: the case where $\mathbf{q}$ is subcritical, and the case where it is critical. We start with the first one. Then by the results of Section 7.1, the walk X satisfies $\mathbb{E}\left[|X_1|\right] < +\infty$ and $\mathbb{E}\left[X_1\right] > 0$, so the conditioning to stay positive is non degenerate. Therefore, it is enough to prove

$$\lim_{C \to +\infty} \liminf_{p \to +\infty} \mathbb{P}\left(X \text{ hits } [p, p+C]\right) = 1. \tag{7.4.3}$$

This follows from standard renewal arguments: if we denote by $(H_i)_{i \geqslant 0}$ the ascending ladder heights of P, then (H_i) is a renewal set with density $\mathbb{E}[H_1] = \frac{1}{\mathbb{E}[X_1]}$. Let I_p be such that $H_{I_p} < p \leqslant H_{I_{p+1}}$. Then the law of $H_{I_{p+1}} - H_{I_p}$ converges as $p \to +\infty$ to the law of H_1 biased by its size, so

$$\mathbb{P}\left(P \text{ does not hit } [p,p+C]\right) \leqslant \mathbb{P}\left(H_{I_{p+1}} \notin [p,p+C]\right)$$
$$\leqslant \mathbb{P}\left(H_{I_{p+1}} - H_{I_p} > C\right) \xrightarrow[p\to+\infty]{} \frac{\mathbb{E}\left[H_1 \mathbb{1}_{H_1>C}\right]}{\mathbb{E}[H_1]},$$

and this last quantity goes to 0 as $C \to +\infty$.

We now tackle the case where $\mathbf{q}$ is critical. This is more complicated since renewal arguments are not available anymore, and the conditioning is now degenerate, so absolute continuity arguments become more elaborate. On the other hand, the growth is now slower and X is recurrent, so it seems more difficult to jump over a large interval. And indeed, we will prove

$$\lim_{p\to+\infty} \mathbb{P}\left(P \text{ hits } p\right) = 1,$$

which is a much stronger version of (7.4.3).

For this, our strategy will be the following: let τ_p be the first time at which P is at least p,

- The scaling limit of P is a process with no positive jump, therefore $P_{\tau_p} = p + o(p)$ as $p \to +\infty$.
- Between time τ_p and $\tau_p + o(p^{2/3})$, the process P looks a lot like a nonconditioned random walk X started from P_{τ_p}.
- If X is started from $p+o(p)$, the time it takes to first hit P is $o(p^{2/3})$. This is a stronger version of the recurrence of X, and will follow from a local limit theorem for random walks.

Let us now be more precise. By Theorem 3 of [Bud15] (see also [Cur19, Chapter 10]), we have the convergence

$$\left(\frac{P_{nt}}{n^{2/3}}\right)_{t\geqslant 0} \xrightarrow[n\to+\infty]{(d)} \left(c_{\mathbf{q}} S_t^+\right)_{t\geqslant 0}$$

for the Skorokhod topology, where S^+ is a 3/2-Lévy process with no positive jump conditioned to stay positive, and $c_{\mathbf{q}}$ is a constant depending on $\mathbf{q}$ that will not be important here. In particular, this process has no positive jump, which implies that $P_{\tau_p} - p = o(p)$ in probability. Hence, there is a deterministic function $f(p)$ with $\frac{f(p)}{p} \to 0$ such that, for any $\varepsilon > 0$,

$$\mathbb{P}\left(P_{\tau_p} - p \geqslant \varepsilon f(p)\right) \xrightarrow[p\to+\infty]{} 0.$$

We now fix $\varepsilon > 0$, and condition on $P_{\tau_p} = p'$ for some $p \leqslant p' \leqslant p+\varepsilon f(p)$. We claim that then $\left(P_{\tau_p+i} - p'\right)_{0 \leqslant i \leqslant f(p)^{3/2}}$ can be coupled with $(X_i)_{0 \leqslant i \leqslant f(p)^{3/2}}$ in such a way that both processes are the same with probability $1 - o(1)$. For this, recall that P can be described as an h-transform of X, where h is given by (7.1.7). Hence, the Radon–Nikodym derivative of the first process with respect to the second is

$$\frac{h_1(p' + X_{f(p)^{3/2}})}{h_1(p')}. \tag{7.4.4}$$

Since $\frac{X_{f(p)^{3/2}}}{f(p)}$ converges in distribution, we have $\frac{X_{f(p)^{3/2}}}{p} \to 0$ in probability. By using the fact that $\frac{p'}{p} \to 0$ uniformly in p' and that $h_1(x) \sim c\sqrt{x}$ for some $c > 0$ (see Section 7.1). , we conclude that (7.4.4) goes to 1 as $p \to +\infty$, uniformly in $p' \in [p, p+\varepsilon f(p)]$. This proves our coupling claim. Note that under this coupling, the time where P hits exactly p is τ_p plus the time where X hits $p - p'$.

We will now show that, if p is large enough, for any $k \in [-\varepsilon f(p), 0]$, we have

$$\mathbb{P}\left(X \text{ hits } k \text{ before time } f(p)^{3/2}\right) \geqslant 1 - \delta(\varepsilon) - o(1), \tag{7.4.5}$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Together with our coupling result, this will imply that the probability for P to hit p before time $\tau_p + f(p)^{3/2}$ is at least $1 - g(\varepsilon) - o(1)$. Since this is true for any $\varepsilon > 0$, this will conclude the proof of Lemma 7.4.6 in the critical case.

The proof of (7.4.5) relies on the Local Limit Theorem (this is e.g. Theorem 4.2.1 of [IL71]). This theorem (in the case $\alpha = 3/2$) states that

$$\sup_{k\in\mathbb{Z}} \left| n^{2/3}\mathbb{P}(X_n = k) - g\left(\frac{k}{n^{2/3}}\right)\right| \xrightarrow[n\to+\infty]{} 0,$$

where g is a continuous function (the density of a 3/2-stable variable). On the other hand, let us denote $t = f(p)^{3/2}$. By the strong Markov property, we have

$$\begin{aligned}\mathbb{E}_0\left[\sum_{i=0}^{t}\mathbb{1}_{X_i=k}\right] &\leqslant \mathbb{P}_0\left(X \text{ hits } k \text{ before time } t\right)\mathbb{E}_k\left[\sum_{i=0}^{t}\mathbb{1}_{X_i=k}\right]\\ &= \mathbb{P}_0\left(X \text{ hits } k \text{ before time } t\right)\mathbb{E}_0\left[\sum_{i=0}^{t}\mathbb{1}_{X_i=0}\right].\end{aligned}$$

Therefore, using the local limit theorem, we can write, for $-\varepsilon f(p) \leqslant k \leqslant 0$ and p large (the o terms are all uniform in k):

$$\begin{aligned}\mathbb{P}_0\left(X \text{ hits } p \text{ before } t\right) &\geqslant \frac{\sum_{i=0}^{t}\mathbb{P}_0\left(X_i=k\right)}{\sum_{i=0}^{t}\mathbb{P}_0\left(X_i=0\right)}\\ &= \frac{\sum_{i=1}^{t}\left(\frac{1}{i^{2/3}}g\left(\frac{k}{i^{2/3}}\right)+o\left(\frac{1}{i^{2/3}}\right)\right)}{1+\sum_{i=1}^{t}\left(\frac{1}{i^{2/3}}g(0)+o\left(\frac{1}{i^{2/3}}\right)\right)}\\ &\geqslant \frac{-\varepsilon t^{1/3}+\sum_{i=\varepsilon t}^{t}\left(\frac{1}{i^{2/3}}\min_{[-\varepsilon^{1/3},0]}g+o\left(\frac{1}{i^{2/3}}\right)\right)}{3t^{1/3}g(0)+\varepsilon t^{1/3}}\\ &\geqslant \frac{-2\varepsilon t^{1/3}+\left(3t^{1/3}-3\varepsilon^{1/3}t^{1/3}\right)\min_{[-\varepsilon^{1/3},0]}g}{\left(3g(0)+\varepsilon\right)t^{1/3}}\\ &= \frac{-2\varepsilon+3(1-\varepsilon^{1/3})\min_{[-\varepsilon^{1/3},0]}g}{3g(0)+\varepsilon},\end{aligned}$$

where the third line uses that, for any index $i \geqslant \varepsilon t$, we have

$$0 \geqslant \frac{k}{i^{2/3}} \geqslant -\frac{\varepsilon f(p)}{(\varepsilon t)^{2/3}} = \varepsilon^{1/3}.$$

We obtain a lower bound that goes to 1 as $\varepsilon \to 0$, so this proves (7.4.5), and Lemma 7.4.6. □

Proof of Proposition 7.4.5. The slight subtlety in the proof is that in the finite world (i.e. on the finite map $\mathbf{M}_n$), we cannot condition on the values of $\mathbf{Q}^1$ and $\mathbf{Q}^2$. However, we can condition on the maps explored at the time where the perimeter of the explored part hits $[p, p+C]$ for the first time. Then Proposition 7.1.2 tells us that this conditioning gives us a good approximation of $\mathbf{Q}^1$ and $\mathbf{Q}^2$.

For $i \in \{1,2\}$, we will denote by $\mathcal{E}_t^{n,i}$ the explored map at time t when we perform a peeling exploration of the map $\mathbf{M}_n$ until time t around the root e_n^i. We will also denote by $\mathcal{E}_t^{\infty,i}$ the explored part in the limit infinite map $\mathbf{M}^i_{\mathbf{Q}^i}$.

We fix a function f among those appearing in the statement of Proposition 7.4.5, and some $\varepsilon > 0$. By Lemma 7.4.6, let C be a constant such that

$$\liminf_{p\to+\infty} \mathbb{P}\left(\sigma^{1,\infty}_{[p,p+C]} < +\infty\right) > 1 - \frac{\varepsilon}{20},$$

and note that C depends only on ε. For p large enough (where "large enough" may depend on ε), there is $v_0(p)$ such that

$$\mathbb{P}\left(\sigma^{\infty,1}_{[p,p+C]} \leqslant v_0 \text{ and } \left|\mathcal{E}^{\infty,1}_{\sigma^{\infty,1}_{[p,p+C]}}\right| \leqslant v_0\right) > 1 - \frac{\varepsilon}{20}. \tag{7.4.6}$$

On the other hand, Proposition 7.1.2 provides a function $\widetilde{f}_j$ on the set of finite maps with a hole such that $\widetilde{f}_j(\mathcal{E}^{\infty,1}_t) \to r_j(\mathbf{Q}^1)$ almost surely as $t \to +\infty$. Let $\eta < 1$ be a small constant, which will be fixed later and will only depend on ε. For p large enough, we have

$$\mathbb{P}\left(\sigma^{\infty,1}_{[p,p+C]} \leqslant v_0 \text{ but } \left|\widetilde{f}_j\left(\mathcal{E}^{\infty,1}_{\sigma^{\infty,1}_{[p,p+C]}}\right) - r_j(\mathbf{Q}^1)\right| \geqslant \frac{\varepsilon}{8}\right) < \eta\frac{\varepsilon}{20}. \tag{7.4.7}$$

From now on, we take p large enough so that both (7.4.6) and (7.4.7) hold. By local convergence and (7.4.6), for n large enough (where "large enough" may depend on ε and p), we have

$$\mathbb{P}\left(\sigma^{n,1}_{[p,p+C]}, \sigma^{n,2}_{[p,p+C]} \leqslant v_0 \text{ and } \left|\mathcal{E}^{n,1}_{\sigma^{n,1}_{[p,p+C]}}\right|, \left|\mathcal{E}^{n,2}_{\sigma^{n,2}_{[p,p+C]}}\right| \leqslant v_0\right) > 1 - \frac{\varepsilon}{10}.$$

By the assumption that $|r_j(\mathbf{Q}^1) - r_j(\mathbf{Q}^2)| > \varepsilon$ with probability at least ε and by (7.4.7), we deduce that

$$\mathbb{P}\left(\begin{array}{c} \sigma^{n,1}_{[p,p+C]}, \sigma^{n,2}_{[p,p+C]} \leqslant v_0 \text{ and } \left|\mathcal{E}^{n,1}_{\sigma^{n,1}_{[p,p+C]}}\right|, \left|\mathcal{E}^{n,2}_{\sigma^{n,2}_{[p,p+C]}}\right| \leqslant v_0 \\ \text{and } \left|\widetilde{f}_j\left(\mathcal{E}^{\infty,1}_{\sigma^{\infty,1}_{[p,p+C]}}\right) - \widetilde{f}_j\left(\mathcal{E}^{\infty,2}_{\sigma^{\infty,2}_{[p,p+C]}}\right)\right| \geqslant \frac{3}{4}\varepsilon \end{array}\right) > \frac{4}{5}\varepsilon.$$

Note that if this last event occurs but the two regions $\mathcal{E}^{n,1}_{\sigma^{n,1}_{[p,p+C]}}$ and $\mathcal{E}^{n,2}_{\sigma^{n,2}_{[p,p+C]}}$ have a common face, then the dual graph distance between the two roots is bounded by $2v_0$. However, by Proposition 7.2.1, the volume of the ball of radius $2v_0$ around e^1_n is tight as $n \to +\infty$, so the probability that this happens goes to 0 as $n \to +\infty$. Hence, for n large enough:

$$\mathbb{P}\left(\begin{array}{c} \mathcal{E}^{n,1}_{\sigma^{n,1}_{[p,p+C]}}, \mathcal{E}^{n,2}_{\sigma^{n,2}_{[p,p+C]}} \text{ are well-defined, face-disjoint, have volume} \\ \text{at most } v_0 \text{ and } \left|\widetilde{f}_j\left(\mathcal{E}^{\infty,1}_{\sigma^{\infty,1}_{[p,p+C]}}\right) - \widetilde{f}_j\left(\mathcal{E}^{\infty,2}_{\sigma^{\infty,2}_{[p,p+C]}}\right)\right| \geqslant \frac{3}{4}\varepsilon \end{array}\right) > 1 - \frac{\varepsilon}{10}. \tag{7.4.8}$$

Now assume that this last event occurs and condition on the σ-algebra $\mathcal{F}_\sigma$ generated by the pair $\left(\mathcal{E}^{n,1}_{\sigma^{n,1}_{[p,p+C]}}, \mathcal{E}^{n,2}_{\sigma^{n,2}_{[p,p+C]}}\right)$ of explored regions. Then, let $I_1, I_2 \in [0, C]$ be such that the perimeters of the two explored regions are $2p + 2I_1$ and $2p+2I_2$. Then the complementary map is a uniform map with genus g_n, two boundaries of length $\left|\partial\mathcal{E}^{n,1}_{\sigma^{n,1}_{[p,p+C]}}\right|$ and $\left|\partial\mathcal{E}^{n,2}_{\sigma^{n,2}_{[p,p+C]}}\right|$, and face degrees $\widetilde{f}_i$ corresponding to the faces of $\mathbf{M}_n$ minus the faces of the two explored regions. More precisely, let F_i be the number of faces of degree $2i$ in $\mathcal{E}^{n,1}_{\sigma^{n,1}_{[p,p+C]}} \cup \mathcal{E}^{n,2}_{\sigma^{n,2}_{[p,p+C]}}$. Then $\widetilde{f}_i = f_i - F_i$. We now perform I_1 peeling steps around $\mathcal{E}^{n,1}_{\sigma^{n,1}_{[p,p+C]}}$, followed by I_2 peeling steps around $\mathcal{E}^{n,2}_{\sigma^{n,2}_{[p,p+C]}}$. We call a peeling step *nice* if it consists of gluing together two boundary edges, which decreases the perimeter by 2. The number of possible values of the map $\mathbf{M}_n \backslash \left(\mathcal{E}^{n,1}_{\sigma^{n,1}_{[p,p+C]}} \cup \mathcal{E}^{n,2}_{\sigma^{n,2}_{[p,p+C]}}\right)$ is

$$\beta_g^{(p+I_1,p+I_2)}(\widetilde{\mathbf{f}}).$$

We count doubly rooted maps since we need to specify how we glue $\mathcal{E}^{n,2}_{\sigma^{n,2}_{[p,p+C]}}$. On the other hand, if the $I_1 + I_2$ additional peeling steps are all good and the regions around e_n^1 and e_n^2 are still disjoint after these steps, the number of possible complementary maps is

$$\beta_g^{(p,p)}(\widetilde{\mathbf{f}}).$$

It follows that

$$\mathbb{P}\left(\text{the } I_1 + I_2 \text{ peeling steps are all nice} | \mathcal{F}_\sigma\right) = \frac{\beta_g^{(p,p)}(\widetilde{\mathbf{f}})}{\beta_g^{(p+I_1,p+I_2)}(\widetilde{\mathbf{f}})}.$$

By the bounded ratio lemma (more precisely, by Lemma 7.2.4), this is always larger than a constant η depending on $\mathbf{f}$ and on ε. More precisely η may depend on I_1 and I_2, but $0 \leqslant I_1, I_2 \leqslant C(\varepsilon)$, so (I_1, I_2) can take finitely many values given ε, so η only depends on ε. This is the value of η that we choose for (7.4.7). For $i \in \{1, 2\}$, we write $\tau_p^{n,i} = \sigma^{n,i}_{[p,p+C]} + I_i$. It follows from this last computation and from (7.4.8) that, for n large enough, we have

$$\mathbb{P}\left(\begin{array}{c} \mathcal{E}^{1,n}_{\tau_p^{n,i}} \text{ and } \mathcal{E}^{2,n}_{\tau_p^{n,2}} \text{ are both face-disjoint, have perimeter } p \\ \text{and volume } \leqslant v_0, \text{ and } \left|\widetilde{f}_j\left(\mathcal{E}^{\infty,1}_{\sigma^{\infty,1}_{[p,p+C]}}\right) - \widetilde{f}_j\left(\mathcal{E}^{\infty,2}_{\sigma^{\infty,2}_{[p,p+C]}}\right)\right| \geqslant \frac{3}{4}\varepsilon \end{array}\right) \geqslant \frac{4}{5}\varepsilon\eta.$$

Finally, we can use (7.4.7) to replace back the approximations $\widetilde{f}_j\left(\mathcal{E}^{\infty,i}_{\sigma^{\infty,i}_{[p,p+C]}}\right)$ by $r_j(\mathbf{Q}^i)$. We obtain

$$\mathbb{P}\left(\begin{array}{c}\mathcal{E}^{1,n}_{\tau_p^{n,i}} \text{ and } \mathcal{E}^{2,n}_{\tau_p^{n,2}} \text{ are both face-disjoint, have perimeter } p \\ \text{and volume } \leqslant v_0, \text{ and } |r_j(\mathbf{Q}^1)-r_j(\mathbf{Q}^2)| \geqslant \frac{1}{2}\varepsilon\end{array}\right) \geqslant \frac{3}{5}\varepsilon\eta.$$

On this event, we have $(\mathbf{M}_n, e_n^1, e_n^2) \in \mathscr{E}_{n,p,v_0}$. Therefore, this concludes the proof of the proposition, with $\delta = \frac{3}{5}\eta\varepsilon$. □

7.4.3 Proof of Theorem 7.4.3

The proof of Theorem 7.4.3 is now basically the same as the two-holes argument of Chapter 6. Therefore, we will not write the argument in full details. We stress right now two differences:

- The first one is that the involution obtained by (possibly) swapping the two explored parts is now non-identity on a relatively small set of maps[4]. The only consequence is that in the end, instead of contradicting the almost sure convergence of Proposition 7.1.2 on an event of probability ε, we will contradict it on an event of probability $\delta < \varepsilon$.
- The other difference is that in Chapter 6, the only observable we were using to approximate the Boltzmann weights was the ratio between perimeter and volume, which corresponds to our function r_∞ here. To deal with the functions r_j for $j \in \mathbb{N}^*$, we simply need the observation that, if $q >> p$, the proportion of peeling steps where we discover a new face of perimeter $2j$ only depends on the part of the exploration after τ_p.

Sketch of proof of Theorem 7.4.3. Let $\varepsilon > 0$, and assume

$$\mathbb{P}\left(|f(\mathbf{Q}^1) - f(\mathbf{Q}^2)| > \varepsilon\right) > \varepsilon.$$

Let $\delta > 0$ be given by Proposition 7.4.5. Consider $p >> 1$ (depending on ε), and define an involution Φ on the set of maps of size n with prescribed genus and face degrees: if $m \in \mathscr{E}_{n,p,v_0}$, then $\Phi(m)$ is obtained from m by swapping the two regions $\mathcal{E}^{1,n}_{\tau_p^{1,n}}$ and $\mathcal{E}^{2,n}_{\tau_p^{2,n}}$. If $m \notin \mathscr{E}_{n,p,v_0}$, then $\Phi(m) = m$. Note that $\Phi(\mathbf{M}_n, e_n^1, e_n^2)$ is still uniform on bi-rooted maps with prescribed genus and face degrees. This map rooted at e_1^n converges to a map $\widehat{\mathbf{M}}$, which has to

[4] i.e., for many maps, there is no swapping.

be a mixture of Boltzmann infinite planar maps. On the other hand, by its construction, $\widehat{M}$ can be described as follows. If we explore $\widehat{\mathbf{M}}$ by the same peeling algorithm as usual and denote by $\widehat{\mathcal{E}}_t$ the explored part at time t, then either

$$\widehat{M} = \mathbf{M}^1_{\mathbf{Q}^1}$$

or

$$\widehat{\mathcal{E}}_{\tau_p^1} = \mathcal{E}^1_{\tau_p^1} \text{ and } \widehat{\mathbf{M}} \backslash \widehat{\mathcal{E}}_{\tau_p^1} = M^2_{\mathbf{Q}^2} \backslash \mathcal{E}^2_{\tau_p^2}.$$

Moreover, by Proposition 7.4.5, if p has been chosen large enough, then with probability at least δ, we are in the second case and $|r_j(\mathbf{Q}^1) - r_j(\mathbf{Q}^2)| > \frac{\varepsilon}{2}$. Now assume $q >> p >> 1$ and assume this event occurs. Then we have

$$\widetilde{f}_j\left(\widehat{\mathcal{E}}_{\tau_p^1}\right) \approx f(\mathbf{Q}^1) \text{ but } \widetilde{f}_j\left(\widehat{\mathcal{E}}_{\widehat{\tau}_q}\right) \approx f(\mathbf{Q}^2), \tag{7.4.9}$$

where $\widehat{\tau}_q$ is the first step where the perimeter of the explored part of $\widehat{\mathbf{M}}$ is at least q. The approximations of (7.4.9) can be made arbitrarily precise if p and q were chosen large enough, so for p large enough and q large enough (depending on p), we have

$$\mathbb{P}\left(\left|\widetilde{f}_j\left(\widehat{\mathcal{E}}_{\tau_p^1}\right) - \widetilde{f}_j\left(\widehat{\mathcal{E}}_{\widehat{\tau}_q}\right)\right| > \frac{\varepsilon}{4}\right) \geqslant \delta,$$

which contradicts the almost sure convergence of Proposition 7.1.2. $\square$

7.4.4 Proof of Corollary 7.4.4

The proof is basically the same as the proof of the main theorem in Chapter 6. The only difference is that we could not prove directly that $d(\mathbf{q})$ and $a_j(\mathbf{q})$ for $j \geqslant 1$ are sufficient to characterize the weight sequence $\mathbf{q}$, so we do not get immediately the uniqueness of the subsequential limit.

More precisely, by the Euler formula, any map with genus g_n and f_i^n faces of degree $2j$ for each j has exactly n edges and $n - \sum_{j\geqslant 1} f_j^n + 2 - 2g_n$ vertices, so, by invariance of $\mathbf{M}_n$ under uniform rerooting, we have

$$\mathbb{E}\left[\frac{1}{\deg_{\mathbf{M}_n}(\rho)}\right] = \frac{n - \sum_{j\geqslant 1} f_j^n + 2 - 2g_n}{2n} \xrightarrow[n\to+\infty]{} \frac{1}{2}\left(1 - 2\theta - \sum_j \alpha_j\right).$$

By the exact same argument as in Chapter 6, we deduce from Theorem 7.4.3 that $d(\mathbf{Q}) = \frac{1}{2}\left(1 - 2\theta - \sum_j \alpha_j\right)$ a.s.. Similarly, by invariance under rerooting, for all $j \geqslant 1$, we have

$$\frac{1}{j}\mathbb{P}\left(\text{the root face of } \mathbf{M}_n \text{ has degree } 2j\right) = \frac{1}{j} \times \frac{2j f_j^n}{2n} \xrightarrow[n\to+\infty]{} \alpha_j.$$

By the same argument as above, we obtain $a_j(\mathbf{Q}) = \alpha_j$ a.s..

7.4.5 Finishing the proof

To conclude the proof of the main theorem, it is enough to show that the weight sequence $\mathbf{q}$ is completely determined by $(a_j(\mathbf{q}))_{j\geqslant 1}$ and $d(\mathbf{q})$. We fix a sequence $(\alpha_j)_{j\geqslant 1}$ such that $\sum_j j\alpha_j = 1$. We recall from Section 7.1 that the weight sequences $\mathbf{q}$ such that $a_j(\mathbf{q}) = \alpha_j$ for all j form a one-parameter family $\left(\mathbf{q}^{(\omega)}\right)_{\omega\geqslant 1}$ given by

$$q_j^{(\omega)} = \frac{j\alpha_j}{\omega^{j-1}h_j(\omega)}\left(\frac{1-\sum_{i\geqslant 1}\frac{1}{4^{i-1}}\binom{2i-1}{i-1}\frac{i\alpha_i}{\omega^{i-1}h_i(\omega)}}{4}\right)^{j-1}.$$

Therefore, to show our main theorem, it is sufficient to prove the following.

Proposition 7.4.7. *The function $\omega \to d(\mathbf{q}^{(\omega)})$ is strictly decreasing.*

Since we were not able to obtain a direct proof of this result, we will deduce it from Corollary 7.4.4. We start with some more basic properties of the function $\omega \to d(\mathbf{q}^{(\omega)})$.

Lemma 7.4.8.
- *The function $\omega \to d(\mathbf{q}^{(\omega)})$ is continuous on $[1,+\infty)$ and analytic on $(1,+\infty)$.*
- *We have $d(\mathbf{q}^{(\omega)}) > 0$ for all ω and $\lim_{\omega\to+\infty} d(\mathbf{q}^{(\omega)}) = 0$.*
- *We have $d(\mathbf{q}^{(\omega)}) \leqslant 1-\sum_{j\geqslant 1}\alpha_j$ for all $\omega \geqslant 1$, with equality if and only if $\omega = 1$.*

Proof. We start with the continuity statement in the first item. The analyticity is more difficult and proved in the appendix. In particular, it implies continuity on $(1,+\infty)$ so it is sufficient to prove the continuity at $\omega = 1$. By the monotone convergence theorem, the function $\omega \to g_{\mathbf{q}^{(\omega)}}$ is continuous at $\omega = 1$, so $q_j^{(\omega)}$ is continuous at $\omega = 1$ for all j. Therefore, for every finite map m with one hole, we have

$$\mathbb{P}\left(m \subset \mathbb{M}_{\mathbf{q}^{(\omega)}}\right) \xrightarrow[\omega\to 1]{} \mathbb{P}\left(m \subset \mathbb{M}_{\mathbf{q}^{(1)}}\right),$$

so $\mathbb{M}_{\mathbf{q}^{(\omega)}} \to \mathbb{M}_{\mathbf{q}^{(1)}}$ in distribution for the local topology. Since the inverse degree of the root vertex is bounded and continuous for the local topology, the function $\omega \to d(\mathbf{q}^{(\omega)})$ is continuous at 1.

We now prove the second item: $d(\mathbf{q}^{(\omega)}) > 0$ is immediate and $d(\mathbf{q}^{(\omega)}) \to 0$ is equivalent to proving $\deg_{\mathbb{M}_{\mathbf{q}^{(\omega)}}}(\rho) \to +\infty$ in probability when $\omega \to +\infty$. For this, we notice that when $\omega \to +\infty$, we have $h_\omega(i) \to 1$ for all i and

$$\widetilde{\nu}_{\mathbf{q}^{(\omega)}}(i) \xrightarrow[\omega\to+\infty]{} \begin{cases} 0 \text{ if } i \leqslant -1, \\ (i+1)\alpha_{i+1} \text{ if } i \geqslant 0. \end{cases}$$

In other words, the probability of any peeling step swallowing at least one vertex goes to 0 when $\omega \to +\infty$. Therefore, if we perform a peeling exploration where we peel the edge on the right of ρ whenever it is possible, the probability to complete the exploration of the root in less than k steps goes to 0 for all k, so the root degree goes to $+\infty$ in probability.

Finally, let us move on to the third item. Since $\mathbb{M}_{\mathbf{q}}$ is stationary, if we denote by $\mathbb{M}_{\mathbf{q}}^{\mathrm{uni}}$ a map with the law $\mathbb{M}_{\mathbf{q}}$ biased by the inverse of the root vertex degree, then $\mathbb{M}_{\mathbf{q}}$ is unimodular. A simple computation shows that $d(\mathbf{q}^{(\omega)}) \leqslant 1 - \sum_{j\geqslant 1} \alpha_j$ is equivalent to $\mathbb{E}[\kappa_{\mathbb{M}_{\mathbf{q}}^{\mathrm{uni}}}(\rho)] \geqslant 0$, where, if v is a vertex of a map m:

$$\kappa_m(v) = 2\pi - \sum_f \frac{\deg(f)-2}{\deg(f)}\pi,$$

and the sum is over all faces that are incident to v, counted with multiplicity. Moreover, we have equality if and only if $\mathbb{E}[\kappa_{\mathbb{M}_{\mathbf{q}}^{\mathrm{uni}}}(\rho)] = 0$. The fact that $\mathbb{E}[\kappa_{\mathbb{M}_{\mathbf{q}}^{\mathrm{uni}}}(\rho)] \geqslant 0$ is then a consequence of [AHNR18, Theorem 1]. Moreover, [AHNR18] shows the equivalence between 17 definition of hyperbolicity. In particular, we have $\mathbb{E}[\kappa_{\mathbb{M}_{\mathbf{q}}^{\mathrm{uni}}}(\rho)] > 0$ (definition 1 in [AHNR18]) if and only if $p_c < p_u$ for bond percolation on $\mathbb{M}_{\mathbf{q}}^{\mathrm{uni}}$. This is equivalent to $\mathbb{P}(p_c < p_u) > 0$ for bond percolation on $\mathbb{M}_{\mathbf{q}}$, which is equivalent to $\omega > 1$ by [Cur19, Theorem 12.9][5]. $\square$

Roughly speaking, the idea behind the end of the proof is the following observation. We fix $j \geqslant 2$. Let $M_{\mathbf{f},g}$ be a uniform map of $\mathcal{B}_g(\mathbf{f})$. If $\mathbf{f}$ is a face degree sequence with $f_j \geqslant 1$, we write $\mathbf{f}^- = \mathbf{f} - \mathbf{1}$. Then we can describe the

[5] More precisely [Cur19, Theorem 12.9] is about half-plane maps. Here is a way to extend it to full-plane maps: there is a percolation regime on the half-plane version of $\mathbb{M}_{\mathbf{q}}$ such that with positive probability, there are infinitely many infinite clusters. For topological reasons, at most two of them intersect the boundary infinitely many times. Hence, with positive probability, there are two infinite clusters that do not touch the boundary. Since there is a coupling in which the half-plane version of $\mathbb{M}_{\mathbf{q}}$ is included in the full-plane version, we have with positive probability two disjoint infinite clusters in $\mathbb{M}_{\mathbf{q}}$ in a certain bond percolation regime, so $p_c < p_u$ with positive probability.

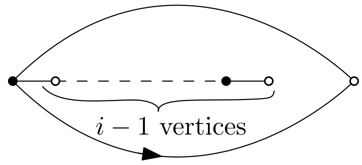


Figure 7.10 – The map m_0^j.

law of $M_{\mathbf{f}^-,g}$ in terms of that of $M_{\mathbf{f},g}$. Indeed $M_{\mathbf{f}^-,g}$ has the law of $M_{\mathbf{f},g} \backslash m_j^0$ conditioned on $m_j^0 \subset M_{\mathbf{f},g}$, where m_j^0 is a deterministic map with a hole of perimeter 2 and only one internal face, which has degree $2j$ (see Figure 7.10).

Therefore, if we know for some suitable face sequences $\mathbf{f}^k$ that $M_{\mathbf{f}^k,g_k}$ converges to an infinite map $\mathbb{M}_{\mathbf{Q}}$, we can deduce that $M_{\mathbf{f}^{k,-},g_k}$ converges to a map $\mathbb{M}_{\widetilde{\mathbf{Q}}}$ and express $\widetilde{\mathbf{Q}}$ in terms of $\mathbf{Q}$. A simple computation (see 7.4.12) shows that $\widetilde{\mathbf{Q}}$ has the law of $\mathbf{Q}$ biased by Q_j. By Lemma 7.1.4, this means that if $\mathbf{Q}$ is not deterministic, then $\widetilde{\mathbf{Q}}$ is "strictly less hyperbolic" than $\mathbf{Q}$ (in the sense that $\omega_{\widetilde{\mathbf{Q}}} < \omega_{\mathbf{Q}}$). On the other hand, the maps $M_{\mathbf{f}^k,g_n}$ and $M_{\mathbf{f}^{k,-},g_n}$ have the same genus but the second is smaller, so it would be natural to expect it to be "more hyperbolic". We will derive a contradiction from this paradox by considering, for a fixed genus, the largest k such that the expected ω corresponding to $M_{\mathbf{f}^k,g_k}$ is smaller than a certain threshold.

Finally, since we want our face degree sequences to respect the proportions (α_j), it makes sense to build the face degree sequences $\mathbf{f}^k$ by adding at each step a face of degree $2j$ with probability proportional to α_j. This is equivalent to taking j random in the above discussion.

More precisely, we build, for every $k \geqslant 0$, a random face-degree sequence $\mathbf{F}^k$ as follows: let $(J_k)_{k\geqslant 1}$ be i.i.d. random variables on $\mathbb{N}^*$ with

$$\mathbb{P}\left(J_k = j\right) = \frac{\alpha_j}{\sum_{i\geqslant 1} \alpha_i}.$$

We write $F_j^k = \sum_{i=1}^k \mathbb{1}_{J_i=j}$, so that $\mathbf{F}^k$ is a face-degree sequence with k faces.

Lemma 7.4.9. *There are variables $\Omega_{k,g}$ such that, whenever $M_{\mathbf{F}^{k_i},g_i}$ converges in distribution to a mixture of infinite Boltzmann maps $\mathbb{M}_{\mathbf{Q}}$, we also have*

$$\Omega_{k_i,g_i} \xrightarrow[k\to+\infty]{(d)} \omega_{\mathbf{Q}}.$$

Proof. Easy: take $t_k \to +\infty$ very slowly, and $\Omega_{k,g}$ is a function of the ratio between the perimeter and volume after t_k peeling steps around the root in $M_{\mathbf{F}^k,g}$. □

Note that the map $M_{\mathbf{F}^k,g}$ is not always well-defined. More precisely, by the Euler formula, this makes sense if and only if $(\mathbf{F}^k, g)$ is *acceptable*, that is

$$\sum_{j \geqslant 1} (j-1) F_j^k \geqslant 2g. \tag{7.4.10}$$

Proof of Proposition 7.4.7. We fix $1 < \omega_0 < +\infty$. For every $g \geqslant 0$, let

$$k_0(g) = \max \left\{ k \geqslant 0 \middle| \mathbb{E}\left[\Omega_{k,g}^{-1}\right] \leqslant \omega_0^{-1} \right\}.$$

The only reason why we take the inverse of $\Omega_{k,g}$ in this definition is to ensure that the expectation is always well-defined. We first claim that, with probability $1 - o(1)$ as $g \to +\infty$, the number $k_0(g)$ is well-defined and the maps $M_{\mathbf{F}^{k_0(g)},g}$ and $M_{\mathbf{F}^{k_0(g)-1},g}$ both exist.

Now fix $g \geqslant 0$. When $k \to +\infty$, the probability that $(\mathbf{F}^k, g)$ is acceptable goes to 1. Moreover, by Corollary 7.4.4 and Lemma 7.4.8 (more precisely the fact that $d(\mathbf{q}) = 1 - \sum_{j \geqslant 1} \alpha_j$ if and only if $\mathbf{q}$ is critical), the local limit of $\mathbb{M}_{\mathbf{F}^k,g}$ as $k \to +\infty$ is the infinite critical Boltzmann map $\mathbb{M}_{\mathbf{q}^{(1)}}$. By Lemma 7.4.9, this implies $\Omega_{k,g} \to 1$ in probability, so $\mathbb{E}[\Omega_{k,g}^{-1}] \to 1$ as $g \to +\infty$. In particular this is larger than ω_0^{-1} for k large enough, so $k_0(g)$ is well-defined and finite.

We now prove that, with probability $1 - o(1)$ as $g \to +\infty$, the pair $(\mathbf{F}^{k_0(g)}, g)$ is acceptable. We first note that by the law of large numbers, we have

$$\frac{1}{k} \sum_{j \geqslant 1} (j-1) F_j^k \xrightarrow[k \to +\infty]{a.s.} \frac{\sum_{j \geqslant 1} (j-1)\alpha_j}{\sum_{j \geqslant 1} \alpha_j} > 0.$$

Let us fix $\varepsilon > 0$ (we will specify its value later), and let

$$k_\varepsilon(g) = (1+\varepsilon) \frac{2 \sum_{j \geqslant 1} \alpha_j}{\sum_{j \geqslant 1} (j-1)\alpha_j} g.$$

Then, with probability $1 - o(1)$ as $g \to +\infty$, the pair $\left(\mathbf{F}^k, g\right)$ is acceptable for all $k \geqslant k_\varepsilon(g)$, so it is enough to prove $k_0(g) \geqslant k_\varepsilon(g)$ with probability $1 - o(1)$. In other words, it is sufficient to prove that, if $\varepsilon > 0$ is well-chosen,

$$\mathbb{E}\left[\Omega_{k_\varepsilon(g),g}^{-1}\right] \leqslant \omega_0^{-1}.$$

For this, up to extracting a subsequence, we can assume $M_{\mathbf{F}^{k_\varepsilon(g)},g} \to \mathbb{M}_{\mathbf{Q}}$ for some random $\mathbf{Q}$. By Corollary 7.4.4 and the fact that $\frac{F_j^k}{|\mathbf{F}^k|} \to \alpha_j$ in probability, the weight sequence $\mathbf{Q}$ is such that $a_j(\mathbf{Q}) = \alpha_j$ for all j, i.e. $\mathbf{Q}$ is of the form $\mathbf{Q} = \mathbf{q}^{(\Omega)}$ for some random variable $\Omega \in [1, +\infty)$. By

Lemma 7.4.9, we also have $\Omega_{k_\varepsilon(g),g} \to \Omega_\varepsilon \in [1, +\infty]$. By the Euler formula, the average inverse degree of the root in $M_{\mathbf{F}^{k_\varepsilon(g)},g}$ is given by

$$\frac{\#V(M_{\mathbf{F}^{k_\varepsilon(g)},g})}{2\#E(M_{\mathbf{F}^{k_\varepsilon(g)},g})} \xrightarrow[g\to+\infty]{} \frac{1}{2}\frac{\varepsilon}{1+\varepsilon}\sum_{j\geqslant 1}(j-1)\alpha_j,$$

so Corollary 7.4.4 again gives

$$d(\mathbf{q}^{\Omega_\varepsilon}) = \frac{1}{2}\frac{\varepsilon}{1+\varepsilon}\sum_{j\geqslant 1}(j-1)\alpha_j \tag{7.4.11}$$

almost surely. Now, since $\omega \to d(\mathbf{q}^{(\omega)})$ is continuous and positive, we can assume that ε was chosen small enough to have

$$\frac{1}{2}\frac{\varepsilon}{1+\varepsilon}\sum_{j\geqslant 1}(j-1)\alpha_j < \min_{1\leqslant\omega\leqslant 2\omega_0} d(\mathbf{q}^{(\omega)}).$$

Then (7.4.11) implies $\Omega_\varepsilon > 2\omega_0$ a.s., so $\mathbb{E}[\Omega_\varepsilon^{-1}] < \frac{\omega_0^{-1}}{2}$, so $\mathbb{E}[\Omega_{k_\varepsilon(g),g}] < \omega_0^{-1}$ for g large enough. We have proved that every subsequence (g_i) has a subsubsequence along which $\mathbb{E}[\Omega_{k_\varepsilon(g),g}] < \omega_0^{-1}$ for g large enough, so $\mathbb{E}[\Omega_{k_\varepsilon(g),g}] < \omega_0^{-1}$ for g large enough. This proves our claim that $M_{\mathbf{F}^{k_0(g)},g}$ is well-defined with probability $1 - o(1)$.

We now know that the maps $M_{\mathbf{F}^{k_0(g)},g}$ and $M_{\mathbf{F}^{k_0(g)+1},g}$ are well-defined with large probability. As before, by 7.4.4 and Lemma 7.4.8, up to extracting a subsequence, we may assume the following joint convergences in distribution as $g \to +\infty$:

$$\begin{aligned}
M_{\mathbf{F}^{k_0(g)},g} &\longrightarrow \mathbb{M}_{\mathbf{q}^{\Omega}},\\
M_{\mathbf{F}^{k_0(g)+1},g} &\longrightarrow \mathbb{M}_{\mathbf{q}^{\widetilde{\Omega}}},\\
\Omega_{k_0(g),g} &\longrightarrow \Omega,\\
\Omega_{k_0(g)+1,g} &\longrightarrow \widetilde{\Omega},\\
\frac{g}{|\mathbf{F}^{k_0(g)}|} &\longrightarrow \theta,
\end{aligned}$$

where $\Omega, \widetilde{\Omega} \in [1, +\infty]$ and $0 \leqslant \theta < \frac{1}{2}\left(1 - \sum_j \alpha_j\right)$.

We now specify the link between the distributions of the maps $M_{\mathbf{F}^{k_0(g)},g}$ and $M_{\mathbf{F}^{k_0(g)+1},g}$. We recall that $\mathbf{F}^{k_0(g)+1} = \mathbf{F}^{k_0(g)} + \mathbf{1}_J$, where $\mathbb{P}(J = j) = \frac{\alpha_j}{\sum_{i\geqslant 1}\alpha_i}$ for all $j \geqslant 1$. If we condition on $\mathbf{F}^{k_0(g)}$ and $\mathbf{F}^{k_0(g)+1}$, then the law of $M_{\mathbf{F}^{k_0(g)},g}$ is the law of $M_{\mathbf{F}^{k_0(g)+1},g} \backslash m_J^0$, conditioned on $m_J^0 \subset M_{\mathbf{F}^{k_0(g)+1},g}$. Therefore, for any map m, we have

$$\mathbb{P}\Big(m \subset M_{\mathbf{F}^{k_0(g)},g} | J=j\Big) = \mathbb{P}\Big(m+m_j^0 \subset M_{\mathbf{F}^{k_0(g)+1},g} | J=j, m_j^0 \subset M_{\mathbf{F}^{k_0(g)+1},g}\Big).$$

By summing over j, we obtain

$$\mathbb{P}\left(m \subset M_{\mathbf{F}^{k_0(g)},g}\right) = \frac{1}{\sum_{i\geqslant 1}\alpha_i}\sum_{j\geqslant 1}\alpha_j \frac{\mathbb{P}\left(m+m_j^0 \subset M_{\mathbf{F}^{k_0(g)+1},g}\right)}{\mathbb{P}\left(m_j^0 \subset M_{\mathbf{F}^{k_0(g)+1},g}\right)}.$$

We now let $g \to +\infty$ to replace $M_{\mathbf{F}^{k_0(g)},g}$ and $M_{\mathbf{F}^{k_0(g)+1},g}$ by respectively $\mathbb{M}_{\mathbf{q}^\Omega}$ and $\mathbb{M}_{\mathbf{q}^{\widetilde{\Omega}}}$. We write p for the perimeter of m, and note that $m_0^j + m$ has the same perimeter as m but one more internal face of degree $2j$. We obtain, for every finite map m with one hole,

$$\mathbb{E}\left[C_p(\mathbf{q}^\Omega)\prod_{f\in m} q^\Omega_{\deg f/2}\right] = \frac{1}{\sum_{i\geqslant 1}\alpha_i}\sum_{j\geqslant 1}\alpha_j \frac{\mathbb{E}\left[C_p(\mathbf{q}^{\widetilde{\Omega}})\times\prod_{f\in m} q^{\widetilde{\Omega}}_{\deg f/2}\times q_j^{\widetilde{\Omega}}\right]}{\mathbb{E}\left[C_1(\mathbf{q}^{\widetilde{\Omega}})q_j^{\widetilde{\Omega}}\right]}.$$

This can be interpreted as a Radon–Nikodym derivative, i.e. the map $\mathbb{M}_{\mathbf{q}^\Omega}$ has the law of $\mathbb{M}_{\mathbf{q}^{\widetilde{\Omega}}}$ biased by

$$\frac{1}{\sum_{i\geqslant 1}\alpha_i}\sum_{j\geqslant 1}\alpha_j \frac{q_j^{\widetilde{\Omega}}}{\mathbb{E}\left[q_j^{\widetilde{\Omega}}\right]} \tag{7.4.12}$$

using the fact that $C_1(\mathbf{q}) = 1$. Since Ω is an almost sure function of the map $\mathbb{M}_{\mathbf{q}^\Omega}$, it follows that Ω has the law of $\widetilde{\Omega}$ biased by (7.4.12). In particular, we have

$$\mathbb{E}\left[\Omega^{-1}\right] = \frac{\sum_{j\geqslant 1}\alpha_j\mathbb{E}\left[q_j^{\widetilde{\Omega}}\,\widetilde{\Omega}^{-1}\right]}{\sum_{j\geqslant 1}\alpha_j\mathbb{E}\left[q_j^{\widetilde{\Omega}}\right]} \tag{7.4.13}$$

On the other hand, by the definition of $m_0(g)$, we have

$$\mathbb{E}\left[\Omega^{-1}_{k_0(g),g}\right] \leqslant \omega_0^{-1} < \mathbb{E}\left[\Omega^{-1}_{k_0(g)+1,g}\right]$$

so, by letting $g \to +\infty$ and using (7.4.13) to express $\mathbb{E}[\Omega^{-1}]$, we obtain

$$\frac{\sum_{j\geqslant 1}\alpha_j\mathbb{E}\left[q_j^{\widetilde{\Omega}}\,\widetilde{\Omega}^{-1}\right]}{\sum_{j\geqslant 1}\alpha_j\mathbb{E}\left[q_j^{\widetilde{\Omega}}\right]} \leqslant \omega_0^{-1} \leqslant \mathbb{E}\left[\widetilde{\Omega}^{-1}\right]. \tag{7.4.14}$$

In particular, we have

$$\sum_{j\geqslant 1}\alpha_j\mathbb{E}\left[q_j^{\widetilde{\Omega}}\widetilde{\Omega}^{-1}\right]\leqslant\sum_{j\geqslant 1}\alpha_j\mathbb{E}\left[q_j^{\widetilde{\Omega}}\right]\mathbb{E}\left[\widetilde{\Omega}^{-1}\right].$$

On the other hand, Lemma 7.1.4 ensures that $q_j^{\widetilde{\Omega}}$ is a nonincreasing function of $\widetilde{\Omega}$, so

$$\mathbb{E}\left[q_j^{\widetilde{\Omega}}\widetilde{\Omega}^{-1}\right]\geqslant\mathbb{E}\left[q_j^{\widetilde{\Omega}}\right]\mathbb{E}\left[\widetilde{\Omega}^{-1}\right]\tag{7.4.15}$$

for all $j\geqslant 1$, so we must have equality in (7.4.14). Moreover, we know that there is $j\geqslant 2$ such that $\alpha_j>0$, so we must have equality in (7.4.15) for this j. By Lemma 7.1.4, this is only possible if $\widetilde{\Omega}$ is deterministic and, by (7.4.14), this implies $\widetilde{\Omega}=\omega_0$. Moreover, by Corollary 7.4.4, we must have

$$d(\mathbf{q}^{\widetilde{\Omega}})=\frac{1}{2}\left(1-2\theta-\sum_{j\geqslant 1}\alpha_j\right),$$

which implies $\theta=\frac{1}{2}\left(1-\sum_{j\geqslant 1}\alpha_j\right)-2d(\mathbf{q}^{(\omega_0)})$. We have proved that in each subsequence (g_i), there is a subsubsequence along which the convergence

$$\frac{g}{|\mathbf{F}^{k_0(g)}|}\xrightarrow[g\to+\infty]{}\frac{1}{2}\left(1-\sum_{j\geqslant 1}\alpha_j\right)-2d(\mathbf{q}^{(\omega_0)})\tag{7.4.16}$$

in probability holds, so this convergence holds for $g\in\mathbb{N}$.

We can now finish the proof. Fix ω_1 with $1<\omega_0<\omega_1<+\infty$, and define $k_1(g)$ using ω_1 in the same way as $m_0(g)$ is defined using ω_0. Then (7.4.16) holds if we replace $k_0(g)$ by $k_1(g)$ and ω_0 by ω_1. Moreover, we have $\omega_1^{-1}<\omega_0^{-1}$, so by definition $k_1(g)\leqslant k_0(g)$ for all g. Therefore, we have $\frac{g}{|\mathbf{F}^{k_1(g)}|}\geqslant\frac{g}{|\mathbf{F}^{k_0(g)}|}$. Letting $g\to+\infty$ and using (7.4.16) and its analogue for ω_1, we deduce

$$\frac{1}{2}\left(1-\sum_{j\geqslant 1}\alpha_j\right)-2d(\mathbf{q}^{(\omega_1)})\geqslant\frac{1}{2}\left(1-\sum_{j\geqslant 1}\alpha_j\right)-2d(\mathbf{q}^{(\omega_0)}),$$

so $d(\mathbf{q}^{(\omega_1)})\leqslant d(\mathbf{q}^{(\omega_0)})$. We have proved that $\omega\to d(\mathbf{q}^{(\omega)})$ is nonincreasing on $(1,+\infty)$. Since it is analytic on $(1,+\infty)$ and nonconstant (because $\lim_{\omega\to+\infty}d(\mathbf{q}^{(\omega)})=0$), it is strictly decreasing on $(1,+\infty)$. By continuity on $[1,+\infty)$, this function is strictly decreasing on $[1,+\infty)$, which concludes the proof. $\square$

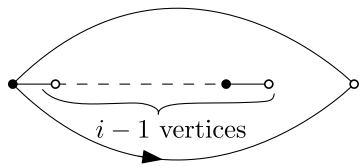


Figure 7.11 – A small piece of map.

7.5 Asymptotic enumeration

In this section, we prove asymptotic results for the number of high genus bipartite maps with prescribed degrees. Let g_n, $\mathbf{f}^{(n)}$, θ and $(\alpha_i)_{i\geqslant 1}$ be defined as in Theorem7.0.1. We will only give the proof of the asymptotics for $\theta > 0$, as in this case there exists a change of variables that shortens the proof and makes it more readable. The proof for $\theta = 0$ works exactly the same, although it is longer to write.

Let $\mathbf{M}_n$ be a uniform map in $\mathcal{B}_{g_n}(\mathbf{f}^{(n)})$. By Theorem 7.0.1, there exists a sequence of numbers $\mathbf{q} = (q_1, q_2, \ldots)$ (with $q_i > 0$ iff $\alpha_i > 0$) such that $\mathbf{M}_n$ converges locally to $\mathbb{M}_{\mathbf{q}}$.

Fix i such that $\alpha_i > 0$. Let m be the map in Figure 7.11. On the one hand, we have

$$\mathbb{P}(m \subset \mathbb{M}_{\mathbf{q}}) = C_2(\mathbf{q})q_i.$$

On the other,

$$\mathbb{P}(m \subset \mathbf{M}_n) = \frac{\beta_{g_n}(\mathbf{f}^{(n)} - \mathbf{1}_i)}{\beta_{g_n}(\mathbf{f}^{(n)})}.$$

The last equality is proven by contracting m in $\mathbf{M}_n$ into a root edge.

By local convergence, we have

$$\frac{\beta_{g_n}(\mathbf{f}^{(n)} - \mathbf{1}_i)}{\beta_{g_n}(\mathbf{f}^{(n)})} \to C_2(\mathbf{q})q_i \tag{7.5.1}$$

as $n \to \infty$. We will make a change of variables to make the calculations easier, and set, for all i,

$$u_i = \frac{i\alpha_i}{\theta},$$

and

$$\phi^i(u_1, u_2, \ldots) := C_2(\mathbf{q})q_i.$$

The function ϕ^i is defined on the domain defined by the inequalities

$$\sum_i u_i > 2 \quad \text{and} \quad \sum_i u_i < \infty. \tag{7.5.2}$$

Lemma 7.5.1. *There exists an integer k and a sequence of functions (u'_i) that are continuous and decreasing such that:*

- $u'_i(\varepsilon) = u_i$ *if* $i > k$,
- $\sum_i u'_i(\varepsilon) = 2 + \varepsilon$,
- *if* $u_i > 0$, *then* $u'_i(\varepsilon) > 0$ *for all* $\varepsilon \geqslant 0$, *and* $u'_i(\varepsilon) \leqslant u_i$ *for* ε *small enough (uniformly).*

Proof. Because $\sum_i u_i < \infty$, there exists k such that

$$r := \sum_{i>k} u_i < 2.$$

Let $c = \sum_{i \leqslant k} u_i$. Set

$$u'_i(\varepsilon) = \frac{2+\varepsilon-r}{c} u_i$$

for $i \leqslant k$ and $u'_i = u_i$ for $i > k$. $\square$

Theorem 7.5.2. *Using the previously defined conventions, we have*

$$\beta_{g_n}(\mathbf{f}^{(n)}) = n^{2g_n} \exp\left(nf(u_1, u_2, \ldots) + o(n)\right), \tag{7.5.3}$$

where

$$f(u_1, u_2, \ldots) = \lim_{\varepsilon \to 0} \Phi(\varepsilon),$$

and

$$\Phi(\varepsilon) = 2\theta(2\log\frac{4}{e} - \log\theta) + \theta\sum_{i=1}^{k}\frac{1}{i}\int_{u_{i'}(\varepsilon)}^{u_i} \log\phi^i(q_1^i(\varepsilon), q_2^i(\varepsilon), \ldots)dt,$$

and $q_i^i(\varepsilon) = t$, $q_j^i(\varepsilon) = u'_i(\varepsilon)$ *for* $j < i$ *and* $q_j^i(\varepsilon) = u_j$ *for* $j > i$.

Let $\mathbf{f}'$ be such that $f'_i = \frac{u'_i(\varepsilon)g}{i}$ for $i \leqslant k$ and $f'_i = f_i$ for $i > k$. The idea of the proof will be to write

$$B_g(\mathbf{f}) = \frac{B_g(\mathbf{f})}{B_g(\mathbf{f}')} \times B_g(\mathbf{f}')$$

for some small $\varepsilon > 0$. The first term will be estimated by an integral via a telescopic product (Proposition 7.5.3), and the second term will be compared to an "extremal case", namely one-faced maps of maximal genus (see Proposition 7.5.4).

By a telescopic product argument similar to the one in Chapter 6 (but in "multiple dimensions" this time), we have:

Proposition 7.5.3.

$$\frac{\beta_g(\mathbf{f})}{\beta_g(\mathbf{f}')} = \exp\left(n\theta \sum_{i=1}^{k} \frac{1}{i} \int_{u_{i'}(\varepsilon)}^{u_i} \log \phi^i(u_1^i(\varepsilon), u_2^i(\varepsilon), \ldots) dt + o(n)\right) \tag{7.5.4}$$

where $u_i^i(\varepsilon) = t$, $u_j^i(\varepsilon) = u_i'(\varepsilon)$ for $j < i$ and $u_j^i(\varepsilon) = u_j$ for $j > i$.

Proof. Let $\mathbf{f}^i$ be defined as $f_j^i = f_j'$ if $j \leqslant i$ and $f_j^i = f_j$ if $j > i$. Note that $\mathbf{f}^0 = \mathbf{f}$ and $\mathbf{f}^k = \mathbf{f}'$. Then we write:

$$\frac{\beta_g(\mathbf{f})}{\beta_g(\mathbf{f}')} = \prod_{i=1}^{k} \frac{\beta_g(\mathbf{f}^{i-1})}{\beta_g(\mathbf{f}^i)}.$$

Now, for all i we have, because of (7.5.1):

$$\log \frac{\beta_g(\mathbf{f}^{i-1})}{\beta_g(\mathbf{f}^i)} = \sum_{t=u_i' g/i}^{u_i g/i} \left(\log \phi^i(u_1', \ldots, u_{i-1}', it/g, u_{i+1}, \ldots) + o(1)\right).$$

If $u_i = 0$, then $f_i^{(n)} = o(n)$, so $\frac{\beta_g(\mathbf{f}^{i-1})}{\beta_g(\mathbf{f}^i)} = \exp(o(n))$ by the Bounded Ratio Lemma.

If $u_i > 0$, the function ϕ^i is bounded below by one, and, because of Lemma 7.2.2 (the bounded ratio lemma) uniformly bounded above by a constant M_ε on the whole region delimited by $\sum_i u_i \geqslant 2+\varepsilon$ and $u_i > u_i(\varepsilon) > 0$. Therefore we can approximate the previous sum by a Riemann sum and we have:

$$\log \frac{\beta_g(\mathbf{f}^{i-1})}{\beta_g(\mathbf{f}^i)} = \frac{g}{i} \int_{u_{i'}(\varepsilon)}^{u_i} \log \phi^i(u_1^i(\varepsilon), u_2^i(\varepsilon), \ldots) dt + o(n).$$

□

Now we will estimate $\beta_g(\mathbf{f}')$.

Proposition 7.5.4. *There is a function h such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and*

$$g^{2g}\left(\frac{4}{e}\right)^{2g} e^{o(g)} \leqslant \beta_g(\mathbf{f}') \leqslant e^{h(\varepsilon)} g^{2g}\left(\frac{4}{e}\right)^{2g} e^{o(g)}.$$

Let $f = \sum_i f_i'$ and $m = \sum_i i f_i'$. Note that $m = (2+\varepsilon)g + o(1)$ and $f \leqslant \varepsilon g + o(1)$ by Euler's formula. Let $U_{g,m}$ be the number of one-faced bipartite maps with m edges. We first need a few lemmas:

Lemma 7.5.5.

$$U_{g,2g+1} = \frac{(4g+1)!!}{g+1} = g^{2g}\left(\frac{4}{e}\right)^{2g} e^{o(g)}.$$

Proof. Adrianov's formula [Adr97] applied to the "extremal" case of one-faced maps with genus g and $2g+1$ edges reads:

$$(2g+2)U_{g,2g+1} = (4g+1)(4g-1)2gU_{g-1,2g-1}.$$

Therefore:

$$U_{g,2g+1} = \frac{(4g+1)!!}{g+1} = g^{2g}\left(\frac{4}{e}\right)^{2g} e^{o(g)}$$

by Stirling's formula. □

Let us introduce some more notations: Let $\mathcal{B}_g(w,b,f)$ be the set of bipartite maps of genus g with w white vertices, b black vertices, and f faces, and $Bip_g(w,b,f)$ its cardinality. We have the following classical lemma, that is a direct consequence of the representation of maps as triples of permutations:

Lemma 7.5.6. *$Bip_g(w,b,f)$ is symmetric in its arguments (w,b,f).*

Lemma 7.5.7. *$Bip_g(w,b,f) \leqslant 4^f\binom{n}{f-1}Bip_g(w,b,1)$ with $n = w+b+f+2g-2$.*

Proof. We adapt a classical argument about tree-rooted maps (going back to [Mul67] in the planar case). Let $\hat{\beta}_g(f)$ be the number of bipartite maps of genus g with f faces and a distinguished spanning tree of the dual (see Figure 7.12 left).

Obviously, $Bip_g(w,b,f) \leqslant \hat{\beta}_g(f)$. Given a map of genus g with f faces, cut its edges that are crossed by the edges of the spanning tree, as in Figure 7.12. We obtain a map with one face, with $n-f+1$ edges, $f-1$ marked white corners, $f-1$ marked black corners. There are $U_{g,n-f+1}$ such maps with one face, $\binom{n}{f-1}$ possible $f-1$-uples of white (resp. black) corners. To reconstruct the map, one also needs to remember how to pair white corners with black corners without any crossings. The number of ways to do so is $Catalan(f-1) < 4^f$. □

Lemma 7.5.8. *Suppose* $\max(w,b,f) < \varepsilon n$ *for $\varepsilon > 0$ small enough. Then there is a function h such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and*

$$Bip_g(w,b,f) \leqslant e^{h(\varepsilon)n+o(n)}U_{g,2g+1}.$$

Proof. A repeated use of Lemmas 7.5.7 and 7.5.6 gives the following inequality:

$$Bip_g(w,b,f) \leqslant 4^{w+b+f}\binom{n}{f-1}\binom{n}{w-1}\binom{n}{w-1}Bip_g(1,1,1).$$

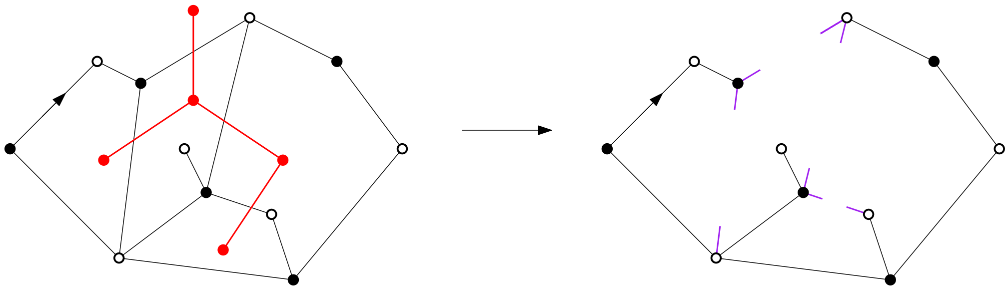


Figure 7.12 – Left: a bipartite map, with a spanning tree of its dual (in red). Right: the "opened" (one-faced) map and the marked corners (in purple).

To finish the proof, we first notice that $Bip_g(1,1,1) = U_{g,2g+1}$. Also, we have the following bound:

$$\binom{n}{f-1} \leqslant \binom{n}{\varepsilon n} \leqslant \exp\Big(n\left(\varepsilon \log \frac{1}{\varepsilon} + (1-\varepsilon)\log\frac{1}{1-\varepsilon}\right) + o(n)\Big)$$

by Stirling's formula. The same bound holds when replacing f by w or b. $\square$

We are now ready to prove Proposition 7.5.4.

Proof of Proposition 7.5.4. The first estimation is quite straightforward.

First, by Lemma 7.2.6, we have the inequality $\beta_g(\mathbf{f}') \geqslant U_{g,m}$. But we also have $U_{g,m} \geqslant U_{g,2g+1}$. We conclude by using Lemma 7.5.5.

For the second inequality, notice that, if we set $v = m+2-2g-f$, then we have $v < \varepsilon g$, and

$$\begin{aligned}
\beta_g(\mathbf{f}') &\leqslant \sum_{w+b=v} Bip_g(w,b,f) \\
&\leqslant \sum_{w+b=v} e^{h(\varepsilon)n+o(n)} U_{g,2g+1} \\
&= e^{h(\varepsilon)n} g^{2g} \left(\frac{4}{e}\right)^{2g} e^{o(g)}
\end{aligned}$$

where in the second line we applied Lemma 7.5.8, and in the third line we applied Lemma 7.5.5. $\square$

We are finally ready to finish the proof:

Proof of Theorem 7.5.2. Let $\omega_n = \frac{1}{n}\log\frac{\beta_g(\mathbf{f})}{n^{2g}}$. To finish the proof, we want to prove that $\Phi(\varepsilon)$ converges as $\varepsilon \to 0$, that ω_n converges as $n \to \infty$, and that both these limits are the same. Propositions 7.5.3 and 7.5.4 combined show that for all $\varepsilon > 0$, we have

$$\Phi(\varepsilon) + o(1) \leqslant \omega_n \leqslant \Phi(\varepsilon) + h(\varepsilon) + o(1) \tag{7.5.5}$$

as $n \to \infty$, knowing that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. By the inequality above, ω_n converges to a finite number ω, and therefore $\Phi(\varepsilon) \to \omega$ as $\varepsilon \to 0$, and the proof is over.

□

7.6 Appendix: proof of the technical lemmas

7.6.1 Proof of Lemma 7.1.4

We recall (7.1.14) here:

$$q_j^{(\omega)} = \frac{\alpha_j}{\omega^{j-1} h_j(\omega)} \left(\frac{1 - \sum_{i \geqslant 1} \frac{1}{4^{i-1}} \binom{2i-1}{i-1} \frac{\alpha_i}{\omega^{i-1} h_i(\omega)}}{4} \right)^{j-1}.$$

Let us write $g_j(\omega) = \frac{\binom{2j-1}{j-1}}{(4\omega)^{j-1} \sum_{i=0}^{j-1} \frac{1}{(4\omega)^i} \binom{2i}{i}}$ (this is a decreasing function of ω). After taking the derivative and splitting the sum, what we want becomes equivalent to the inequality:

$$g_j'(\omega) - g_j'(\omega) g_i(\omega) - (j-1) g_j(\omega) g_i'(\omega) \leqslant 0$$

for all $i, j \geqslant 1$ and $\omega \geqslant 1$.

Let's write $g_j(\omega) = \frac{\binom{2j-1}{j-1}}{P_j}$. We have $P_1 = 1$, $P_0 = 0$ and

$$P_{j+1} = 4\omega P_j + \binom{2j}{j}.$$

The problem rewrites

$$F(i,j) := P_i^2 P_j' - \binom{2i-1}{i-1} \left(P_j' P_i + (j-1) P_i' P_j \right) \geqslant 0.$$

Let us introduce

$$\begin{aligned} \Delta F(i,j) &:= F(i, j+1) - 4\omega F(i,j) \\ &= 4 P_j P_i^2 - \binom{2i-1}{i-1} \left(4 P_j P_i + j \binom{2j}{j} P_i' + 4\omega P_i' P_j \right). \end{aligned}$$

and

$$\begin{aligned}
\Delta^2 F(i,j) &:= \Delta F(i,j+1) - 4\omega \Delta F(i,j) \\
&= 4\binom{2j}{j} P_i^2 - \binom{2i-1}{i-1}\left(4\binom{2j}{j} P_i\right) \\
&\quad - \binom{2i-1}{i-1}\left(((j+1)\binom{2(j+1)}{j+1} - 4\omega(j-1)\binom{2j}{j})P_i'\right) \\
&\geqslant \binom{2j}{j}\left(4P_i^2 - \binom{2i-1}{i-1}(4P_i + 6P_i')\right).
\end{aligned}$$

Let $G(i) := P_i^2 - \binom{2i-1}{i-1}\left(P_i + \frac{3}{2}P_i'\right)$. We want to prove that for $\omega \geqslant 1$, $G(i) \geqslant 0$. On the one hand, since $deg(P_i) = i-1$, for all $k \geqslant i$,

$$[\omega^k]G(i) \geqslant 0.$$

On the other hand, let $k \leqslant i-2$, then

$$\begin{aligned}
[\omega^k]G(i) &\leqslant [\omega^k]P_i^2 - \binom{2i-1}{i-1}P_i' \\
&= \sum_{a+b=k} 4^k \binom{2(i-1-a)}{i-1-a}\binom{2(i-1-b)}{i-1-b} \\
&\quad - 4^{k+1}(k+1)\binom{2i-1)}{i-1}\binom{2(i-2-k)}{i-2-k}.
\end{aligned}$$

It is easily checked that the quantity $\binom{2(i-1-a)}{i-1-a}\binom{2(i-1-b)}{i-1-b}$ is maximal when $a = k$ or $b = k$. Finally, since we have

$$4\binom{2i-1}{i-1}\binom{2(i-2-k)}{i-2-k} \geqslant \binom{2(i-1)}{i-1}\binom{2(i-1-k)}{i-1-k}$$

we can conclude that

$$[\omega^k]G(i) \leqslant 0.$$

In other words, we have proved that the coefficients of $G(i)$ are all positive, then all negative (starting from the highest order coefficients). This implies that $G(i)$ has a unique real root ω^*, and is positive for all $\omega > \omega^*$.

It remains to show that $\omega^* = 1$, i.e. that $G(i) = 0$ when $\omega = 1$. But this is true since it is easily checked that

$$P_i(1) = \frac{i}{2}\binom{2i}{i}$$

and

$$P_i'(1) = \frac{i(i-1)}{3}\binom{2i}{i}.$$

Therefore, for all i and j, we have

$$\Delta^2 F(i,j) \geqslant 0.$$

But since $\Delta F(i,0) = 0$, we have $\Delta F(i,j) \geqslant 0$, and since $F(i,0) = 0$, we have

$$F(i,j) \geqslant 0.$$

7.6.2 The face degree transfer lemmas

Here we prove the face degree transfer lemmas.

Proof of Lemma 7.2.5. This is proved by an injection. Take a map in $\mathcal{B}_g(\mathbf{f} - \mathbf{1}_p)$, pick an edge e that has a face of degree $2i$ lying on its right (there are if_i of those). Glue a path of $p-i$ edges to the black vertex of e (see Figure 7.13). One obtains a map of $\mathcal{B}_g(\mathbf{f} - \mathbf{1}_i)$ with a marked edge. Going backwards is straightforward.

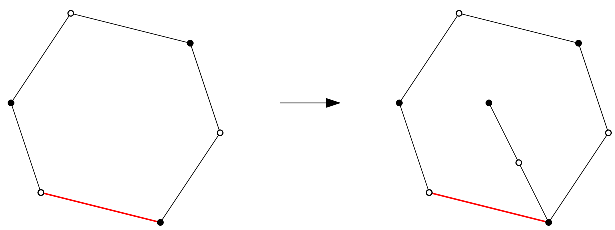


Figure 7.13 – The injection (here with $i = 3$ and $p = 5$). The marked edge is in red.

□

Proof of Lemma 7.2.6. This is again an injection. Take a map in the set $\mathcal{B}_g(\mathbf{f} - \sum_{j=1}^{k} \mathbf{1}_{d_j})$, and pick an edge e that has a face of degree $2i$ lying on its right (there are if_i of those).

Let $d = \sum_{j=1}^{k} d_j - (k-1)$. If $d > i$, transform the face of degree $2i$ into a face of degree $2d$ by adding a path of $d - i$ edges like in the previous proof. Then "tessellate" this face as in Figure 7.14.

One obtains a map of $\mathcal{B}_g(\mathbf{f} - \mathbf{1}_i)$ with a marked edge. Going backwards is straightforward.

□

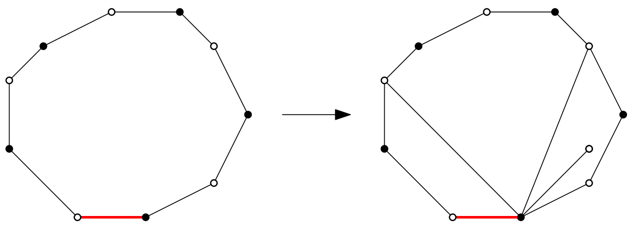


Figure 7.14 – The injection (here with $i = 5$ and $(d_1, d_2, d_3) = (2, 3, 3)$).

7.6.3 The estimation lemmas

Here we prove Lemmas 7.2.12 and 7.2.14. We restate (7.1.3) here:

$$\binom{n+1}{2}\beta_g(\mathbf{f}) = \sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ g_1+g_2+g^*=g}} (1+n_1)\binom{v_2}{2g^*+2}\beta_{g_1}(\mathbf{s})\beta_{g_2}(\mathbf{t}) + \sum_{g^*\geqslant 0}\binom{v+2g^*}{2g^*+2}\beta_{g-g^*}(\mathbf{f}).$$

First, we have the following lemmas.

Lemma 7.6.1. *Under the assumptions of the estimation lemmas, there is a constant C such that:*

$$Cn\beta_g(\mathbf{f}) \geqslant \sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ \mathbf{s},\mathbf{t}\neq\mathbf{0}\\ g_1+g_2=g}} n_1\beta_{g_1}(\mathbf{s})n_2\beta_{g_2}(\mathbf{t})$$

and

$$2C\beta_g(\mathbf{f}) \geqslant \sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ \mathbf{s},\mathbf{t}\neq\mathbf{0}\\ g_1+g_2=g}} \beta_{g_1}(\mathbf{s})\beta_{g_2}(\mathbf{t}).$$

Proof. By only keeping certain terms in (7.1.3), we have:

$$\binom{n+1}{2}\beta_g(\mathbf{f}) \geqslant \sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ \mathbf{s},\mathbf{t}\neq\mathbf{0}\\ g_1+g_2=g\\ v_2\geqslant v_1}} (1+n_1)\binom{v_2}{2}\beta_{g_1}(\mathbf{s})\beta_{g_2}(\mathbf{t}). \tag{7.6.1}$$

Since $v_1 + v_2 = v + 2$, we have

$$v_2 - 1 \geqslant \frac{v}{2} \geqslant \frac{\kappa}{2}n \geqslant \frac{\kappa}{2}n_2$$

in every term of the sum above. Therefore, $\binom{v_2}{2} \geqslant \frac{\kappa^2}{8}nn_2$. We can use the following crude bounds: $\binom{n+1}{2} \leqslant n^2$, $(1+n_1) \geqslant n_1$. By using the last three bounds in (7.6.1), we obtain

$$n^2\beta_g(\mathbf{f}) \geqslant \frac{\kappa^2}{8}\sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ \mathbf{s},\mathbf{t}\neq\mathbf{0}\\ g_1+g_2=g\\ v_2\geqslant v_1}} n_1 n n_2 \beta_{g_1}(\mathbf{s})\beta_{g_2}(\mathbf{t}).$$

By symmetry,

$$\sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ \mathbf{s},\mathbf{t}\neq\mathbf{0}\\ g_1+g_2=g\\ v_2\geqslant v_1}} n_1 n_2 \beta_{g_1}(\mathbf{s})\beta_{g_2}(\mathbf{t}) \geqslant \frac{1}{2}\sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ \mathbf{s},\mathbf{t}\neq\mathbf{0}\\ g_1+g_2=g}} n_1 n_2 \beta_{g_1}(\mathbf{s})\beta_{g_2}(\mathbf{t}),$$

so the first inequality of the Lemma is proved.

The second inequality follows from the first and the fact that in every term of the summand, we have $n_1n_2 \geqslant \frac{n}{2}$ (since $n_1+n_2=n$). $\square$

We also have this Lemma (whose proof is straightforward from (7.1.3), by keeping one term in the second sum of the RHS):

Lemma 7.6.2. *Under the assumption $v \geqslant \kappa n$, we have*

$$\beta_{g'}(\mathbf{f}) = O(n^{-2(g-g')}\beta_g(\mathbf{f}))$$

as $n\to\infty$.

Proof of Lemma 7.2.12. Let i^* be the smallest $i>1$ such that $\frac{if_i}{n}>\delta$ for n large enough. By tessellating the boundary by faces of degree i^*,

$$\beta_{g_j}^{(p_1^j,p_2^j,\ldots,p_{\ell_j}^j)}(\mathbf{h}^{(j)}) \leqslant n^{\ell_j-1}\beta_{g_j}(\mathbf{h'}^{(j)})$$

where $\mathbf{h'}^{(j)} = \mathbf{h}^{(j)} + (\sum_{i=1}^{\ell_j} p_i^j)\mathbf{1}_{i^*}$. That is because each boundary of size p can be tessellated by at most p faces of size i^*.

Therefore,

$$\prod_{j=1}^{k}\beta_{g_j}^{(p_1^j,p_2^j,\ldots,p_{\ell_j}^j)}(\mathbf{h}^{(j)}) \leqslant n^{\sum_{i=1}^{k}(\ell_j-1)}\prod_{j=1}^{k}\beta_{g_j}(\mathbf{h'}^{(j)}).$$

Moreover, by Lemma 7.6.1 there is a constant C' s.t.:

$$\sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ \mathbf{s},\mathbf{t}\neq\mathbf{0}\\ g_1+g_2=g}} \beta_{g_1}(\mathbf{s})\beta_{g_2}(\mathbf{t}) \leqslant C'\beta_g(\mathbf{f}),$$

so, by an easy induction

$$\sum_{\substack{\mathbf{h}^{(1)}+\mathbf{h}^{(2)}+\ldots+\mathbf{h}^{(k)}=\mathbf{h} \\ g_1+g_2+\ldots+g_k=g-1-\sum_j(\ell_j-1)}} \prod_{j=1}^{k} \beta_{g_j}(\mathbf{h}'^{(j)}) \leqslant C''\beta_{g'}(\mathbf{h}')$$

with $\mathbf{h}' = \mathbf{h}'^{(1)} + \mathbf{h}'^{(2)} + \ldots + \mathbf{h}'^{(k)}$, C'' another constant and

$$g' = g_1 + g_2 + \ldots + g_k = g - 1 - \sum_j (\ell_j - 1).$$

We have $\mathbf{h}' = \mathbf{h} + p\mathbf{1}_{i^*}$ where

$$p = \sum_{j=1}^{k} \sum_{i=1}^{\ell_j} p_i^j.$$

Since $\mathbf{h} \leqslant \mathbf{f}$, we have

$$\beta_{g'}(\mathbf{h}') \leqslant \beta_{g'}(\mathbf{f} + p\mathbf{1}_{i^*}) \leqslant C'''\beta_{g'}(\mathbf{f})$$

where in the end we used the bounded ration lemma.

We just bounded the LHS by $Cn^{g-g'-1}\beta_{g'}(\mathbf{f})$ where C is a constant. And finally, by Lemma 7.6.2,

$$\beta_{g'}(\mathbf{f}) = O(n^{-2(g-g')}\beta_g(\mathbf{f}))$$

which is enough to finish the proof. □

Proof of Lemma 7.2.14. The proof of the first part works exactly the same as in the proof of Lemma 7.2.12, except in the end where $g' = g - \sum_j(\ell_j - 1)$ (the -1 is not here anymore), so the LHS is now bounded by $Cn^{g-g'}\beta_{g'}(\mathbf{f})$. Nevertheless, since the ℓ_j's are not all equal to 1, we have $g' < g$ and the conclusion remains the same.

Now we prove the second point. First, by tessellating the boundary, we have

$$\beta_{g_j}^{p^j}(\mathbf{h}^{(j)}) \leqslant \beta_{g_j}(\mathbf{h}'^{(j)})$$

where $\mathbf{h}'^{(j)} = \mathbf{h}^{(j)} + p^j\mathbf{1}_{i^*}$.

By Lemma 7.6.1, we have the bound

$$\sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f} \\ \mathbf{s},\mathbf{t}\neq\mathbf{0} \\ g_1+g_2=g}} n_1\beta_{g_1}(\mathbf{s})n_2\beta_{g_2}(\mathbf{t}) \leqslant C'n\beta_g(\mathbf{f})$$

for a certain constant C'. By induction (and keeping only certain terms in the sum), we have

$$\sum_{\substack{\mathbf{h}^{(1)}+\mathbf{h}^{(2)}+\ldots+\mathbf{h}^{(k)}=\mathbf{h}\\ g_1+g_2+\ldots+g_k=g\\ n_1,n_2>a}} \prod_{j=1}^{k} n_j \beta_{g_j}(\mathbf{h}'^{(j)}) \leqslant C'' n \beta_g(\mathbf{h}')$$

with $\mathbf{h}' = \mathbf{h}'^{(1)} + \mathbf{h}'^{(2)} + \ldots + \mathbf{h}'^{(k)}$, C'' another constant. In every term of the summand, there must exist j^* such that $n_j^* \geqslant \frac{n-u}{k}$. Thus the product $\prod_{j=1}^{k} n_j$ can be bounded below by $\frac{(n-u)a}{k} > \frac{na}{2k}$ for n large enough. Therefore

$$\sum_{\substack{\mathbf{h}^{(1)}+\mathbf{h}^{(2)}+\ldots+\mathbf{h}^{(k)}=\mathbf{h}\\ g_1+g_2+\ldots+g_k=g\\ n_1,n_2>a}} \prod_{j=1}^{k} \beta_{g_j}(\mathbf{h}'^{(j)}) \leqslant \frac{C''''}{a} n \beta_g(\mathbf{h}').$$

We have $\mathbf{h}' = \mathbf{h} + p\mathbf{1}_{i^*}$ where

$$p = \sum_{j=1}^{k} p^j.$$

Since $\mathbf{h} \leqslant \mathbf{f}$, we have

$$\beta_g(\mathbf{h}') \leqslant \beta_g(\mathbf{f} + p\mathbf{1}_{i^*}) \leqslant C''' \beta_g(\mathbf{f})$$

where in the end we used the bounded ration lemma. We just bounded the LHS by $\frac{C}{a}\beta_g(\mathbf{f})$ where C is a constant, thus, we are done.

$\square$

7.6.4 Proof of the analyticity in Lemma 7.4.8

We fix $(\alpha_j)_{j\geqslant 1}$ such that $\sum_{j\geqslant 1} j\alpha_j = 1$ and recall that, for all $\omega \in [1, +\infty)$, the weight sequence $\mathbf{q}^{(\omega)}$ is given by (7.1.14). More precisely, we have

$$q_j^{(\omega)} = \frac{j\alpha_j}{\omega^{j-1} g(\omega)^{j-1} h_\omega(j)}, \tag{7.6.2}$$

where $h_\omega(i) = \sum_{s=0}^{i-1} \binom{2s}{s} (4\omega)^{-s}$ and

$$g(\omega) = \frac{4}{1 - \sum_{i\geqslant 1} \frac{1}{4^{i-1}} \binom{2i-1}{i-1} \frac{i\alpha_i}{\omega^{i-1} h_\omega(i)}}.$$

If we denote by ρ the root vertex of an infinite map $\mathbb{M}_{\mathbf{q}}$, our goal is to prove that the function

$$\omega \to \mathbb{E}\left[\frac{1}{\deg_{\mathbb{M}_{\mathbf{q}(\omega)}}(\rho)}\right]$$

is analytic on $(1,+\infty)$. The first step of the proof is to make sure that $g(\omega)$ is analytic, which will ensure that each of the q_j is an analytic function of ω. For this, we will prove that for each $\omega \in (1,+\infty)$, the infinite sum in the definition of g converges uniformly on a complex neighbourhood of ω. Since each term of the sum is analytic (as an inverse of a polynomial), this will be enough to conclude.

We fix $\omega_0 > 1$ and write $\delta = \frac{1}{2}(\omega_0 - 1)$. We also fix an integer s_0 which will be specified later. For ω complex with $|\omega-\omega_0| < \delta$, we have $|\omega| \geqslant 1+\frac{\delta}{2}$. Hence, for $i \geqslant s_0$, we can write

$$\left|\sum_{s=s_0+1}^{i-1}\binom{2s}{s}(4\omega)^{-s}\right| \leqslant \sum_{s>s_0}|\omega|^{-s} \leqslant \frac{\left(1+\frac{\delta}{2}\right)^{-s_0}}{\delta/2}.$$

Therefore, we have, for $i \geqslant s_0$, we have:

$$|h_\omega(i)| \geqslant \left|\sum_{s=0}^{s_0}\binom{2s}{s}(4\omega)^{-s}\right| - \frac{2}{\delta}\left(1+\frac{\delta}{2}\right)^{-s_0}. \tag{7.6.3}$$

We now fix the value of s_0: we choose s_0 large enough to have

$$\frac{2}{\delta}\left(1+\frac{\delta}{2}\right)^{-s_0} < \frac{1}{8\omega_0}.$$

We now know that the function $\omega \to \sum_{s=0}^{\min(i,s_0)}\binom{2s}{s}(4\omega)^{-s}$ is continuous and is at least $1+\frac{1}{2\omega_0}$ for $\omega = \omega_0$ (it is a sum of nonnegative terms where the term $s=0$ is 1 and the term $s=1$ is $\frac{1}{2\omega_0}$). Therefore, there is $0<\varepsilon<\delta$ such that, for $|\omega-\omega_0|<\varepsilon$, we have

$$\left|\sum_{s=0}^{s_0}\binom{2s}{s}(4\omega)^{-s}\right| > 1+\frac{1}{4\omega_0}.$$

From now on, we consider ω in this complex ball. By (7.6.3), we have

$$|h_\omega(i)| \geqslant 1+\frac{1}{8\omega_0},$$

for $i > s_0$, so

$$\sup_{\omega \in B(\omega_0,\varepsilon)} \left| \frac{1}{4^{i-1}} \binom{2i-1}{i-1} \frac{i\alpha_i}{\omega^{i-1} h_\omega(i)} \right| \leqslant \frac{1}{4^{i-1}} \binom{2i-1}{i-1} \frac{i\alpha_i}{\omega^{i-1}\left(1+\frac{1}{8\omega_0}\right)},$$

which is summable over i. Therefore, the infinite sum in the definition of g is a uniform limit of analytic functions in a neighbourhood of ω_0, so it is analytic in this neighbourhood. In particular, we know that $g(\omega) > 0$ for $\omega > 1$ real, so there is a complex neighbourhood $\mathcal{D}_1$ of $\{\omega > 1\}$ on which $|g(\omega)| > 0$. In this neighbourhood, all the $q_j^{(\omega)}$ are well-defined by the formula (7.6.2) and analytic in ω.

We will write the expected inverse degree of the root as a sum over all the possible values of the hull of radius 1 of $\mathbb{M}_{\mathbf{q}}$. To control this sum, we will need to make sure that the q_j^ω are not too large, i.e. that they are "admissible" even for ω complex.

Lemma 7.6.3. *Let $\omega_0 > \omega_1 > 1$ be real. There is a complex neighbourhood of ω_0 on which $\left|q_j^{(\omega)}\right| \leqslant q_j^{(\omega_1)}$ for all $j \geqslant 1$. In particular, on a complex neighbourhood of the line $\{\omega > 1\}$, the sequence $\left(\left|q_j^{(\omega)}\right|\right)_{j \geqslant 0}$ is admissible and subcritical.*

Proof. First, the last part of the lemma is an immediate consequence of the first one, since a weight sequence dominated term by term by an admissible and subcritical weight sequence is also admissible and subcritical. We now prove the first part.

For $j = 1$, we have $q_1^{(\omega)} = \alpha_1$ for all ω, so $q_1^{(\omega)} = q_1^{(\omega_1)}$ in particular. We now find a complex neighbourhood of ω_0 on which $\left|q_j^{(\omega)}\right| \leqslant q_j^{(\omega_1)}$ for all $j \geqslant 2$. Let $\delta > 0$ to be specified later. By the same argument as above, there is a complex neighbourhood of ω_0 on which

$$|h_\omega(j)| \geqslant (1-\delta)\,|h_{\omega_0}(j)|$$

for all $j \geqslant 1$. Moreover, we have $\omega_0 g(\omega_0) > 0$, so by continuity there is a complex neighbourhood of ω_0 on which

$$|\omega g(\omega)| \geqslant (1-\delta)\,|\omega_0 g(\omega_0)|\,.$$

By combining the last two equations, we get

$$\left|q_j^{(\omega)}\right| \leqslant \frac{1}{(1-\delta)^j} q_j^{(\omega_0)} = \frac{1}{1-\delta} \left(\frac{\omega_1 g(\omega_1)}{(1-\delta)\omega_0 g(\omega_0)} \right)^{j-1} \frac{h_{\omega_1}(j)}{h_{\omega_0}(j)} \times q_j^{(\omega_1)} \quad (7.6.4)$$

for all $j \geqslant 2$. We know from Lemma 7.1.4 that $q_2^{(\omega)}$ is a decreasing function of ω for ω real. Since $h_{\omega_0}(j)$ is decreasing in ω_0, it means that $\omega \to \omega g(\omega)$ is increasing over ω real, so if δ is chosen small enough, we have $\frac{\omega_1 g(\omega_1)}{(1-\delta)\omega_0 g(\omega_0)} < 1$. Since $\frac{h_{\omega_1}(j)}{h_{\omega_0}(j)} \leqslant C\sqrt{j}$ for some absolute constant C, we conclude by (7.6.4) that, on a complex neighbourhood of ω_0, we have

$$\left|q_j^{(\omega)}\right| < q_j^{(\omega_1)} \tag{7.6.5}$$

for all $j \geqslant j_0$, where j_0 may depend on ω_0 and ω_1 but not on ω. Moreover, for $2 \leqslant j \leqslant j_0$, we know by Lemma 7.1.4 that $q_j^{(\omega_0)} < q_j^{(1)}$ and that $q_j^{(\omega)}$ is continuous in ω, so up to shrinking our neighbourhood of ω_0, (7.6.5) holds for all $j \geqslant 2$. This proves the lemma. $\square$

We now move on to the proof of analyticity. For this, we will write the expected inverse degree of the root as an infinite sum over all possible values of the "ball" of radius 1. For this, we first precise the notion of ball that we will use [6]. We consider the peeling algorithm $\mathcal{A}_{\text{root}}$ such that, if the root vertex ρ is on ∂m, then $\mathcal{A}_{\text{root}}(m)$ is the edge on ∂m on the right of the root vertex. If M is a map, we perform a filled-in peeling exploration of M using the algorithm $\mathcal{A}_{\text{root}}$, and denote it by $\left(\mathcal{E}_n^{\mathcal{A}}(M)\right)_{n \geqslant 0}$. We stop the exploration at the first time τ where $\rho \notin \mathcal{E}_n^{\mathcal{A}}(M)$, which is finite almost surely if ρ has finite degree. We denote by $B_1^\bullet(M)$ the explored map $\mathcal{E}_\tau^{\mathcal{A}}(M)$. Finally, we denote by $\mathcal{H}$ the set of maps m such that there is an infinite map M with $B_1^\bullet(M) = m$. This is an infinite set of finite maps.

We can now write, for $\omega > 1$:

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{\deg_{\mathbb{M}_{\mathbf{q}^{(\omega)}}}(\rho)}\right] &= \sum_{m \in \mathcal{H}} \frac{1}{\deg_m(\rho)} \mathbb{P}\left(B_1^\bullet\left(\mathbb{M}_{\mathbf{q}^{(\omega)}}\right) = m\right) \\
&= \sum_{m \in \mathcal{H}} \frac{1}{\deg_m(\rho)} \mathbb{P}\left(m \subset \mathbb{M}_{\mathbf{q}^{(\omega)}}\right) \\
&= \sum_{m \in \mathcal{H}} \frac{1}{\deg_m(\rho)} \left(\omega g(\omega)\right)^{|\partial m|-1} h_\omega(|\partial m|) \prod_{f \in m} q_{|f|}^{(\omega)},
\end{aligned}$$

where $|\partial m|$ is half of the degree of the hole of m, $|f| = \deg f/2$, and the product is over internal faces f of m. Since $q^{(\omega)}$ is well-defined and analytic in ω on a complex neighbourhood of $\{\omega > 1\}$, each term of the sum is well-defined and analytic in such a neighbourhood. We will now prove the following result:

[6] This notion is actually closer to the *hull* of a map. Since it is not useful here, we will not compare it to other notions of hull introduced in the literature.

Lemma 7.6.4. *Let $\omega_0 > 1$. There is a complex neighbourhood $\mathcal{N}$ of ω_0 such that*

$$\sum_{m\in\mathcal{H}} \sup_{\omega\in\mathcal{N}} \left| (\omega g(\omega))^{|\partial m|-1} h_\omega(|\partial m|) \prod_{f\in m} q_{|f|}^{(\omega)} \right| < +\infty.$$

Once Lemma 7.6.4 is proved, it follows that the sum defining $\mathbb{E}\left[\frac{1}{\deg_{\mathbb{M}_{\mathbf{q}^{(\omega)}}}(\rho)}\right]$ converges uniformly on a complex neighbourhood of ω_0, so the sum is analytic, so $\mathbb{E}\left[\frac{1}{\deg_{\mathbb{M}_{\mathbf{q}^{(\omega)}}}(\rho)}\right]$ is the restriction to $\{\omega > 1\}$ of a complex analytic function, so it is analytic. Therefore, all we have to prove is Lemma 7.6.4.

Proof of Lemma 7.6.4. We write $p_m^{(\omega)} = (\omega g(\omega))^{|\partial m|-1} h_\omega(|\partial m|) \prod_{f\in m} q_{|f|}^{(\omega)}$.

We first replace $q_j^{(\omega)}$ using (7.6.2) to obtain

$$\left|p_m^{(\omega)}\right| = |\omega g(\omega)|^{|\partial m|-1-\sum_{f\in m}(|f|-1)} \times |h_\omega(|\partial m|)| \times \prod_{f\in m} \frac{|f|\alpha_{|f|}}{|h_\omega(|f|)|}.$$

An application of the Euler formula shows that $|\partial m| - 1 - \sum_{f\in m}(|f|-1)$ is equal to minus the number of internal vertices of m, which we denote by $V_{\text{int}}(m) \geqslant 0$. We now fix ω_1, ω_2 real with $1 < \omega_1 < \omega_0 < \omega_2$. Similarly as before, we know that $h_\omega(j)$ (resp. $\omega \to \omega g(\omega)$) is decreasing (resp. increasing) on $(1, +\infty)$ and analytic, therefore there is a neighbourhood of ω_0 in which we have

$$|\omega g(\omega)| \geqslant \omega_1 g(\omega_1) \quad \text{and} \quad \forall j \geqslant 2,\ h_{\omega_2}(j) \leqslant |h_\omega(j)| \leqslant h_{\omega_1}(j) \leqslant \frac{1}{\sqrt{1-\omega_1^{-1}}}$$

where the rightmost inequality comes from the Taylor expansion of $\frac{1}{\sqrt{1-\omega_1^{-1}}}$. In this neighbourhood $\mathcal{N}$, we have

$$\left|p_m^{(\omega)}\right| \leqslant \frac{1}{\sqrt{1-\omega_1^{-1}}} (\omega_1 g(\omega_1))^{-V_{\text{int}}(m)} \prod_{f\in m} \frac{|f|\alpha_{|f|}}{h_{\omega_2}(|f|)}.$$

On the other hand, for $m \in \mathcal{H}$, we have

$$\mathbb{P}\left(B_1^\bullet\left(\mathbb{M}_{\mathbf{q}^{(\omega_2)}}\right) = m\right) = p_m^{(\omega_2)} = h_{\omega_2}(|\partial m|)\,(\omega_2 g(\omega_2))^{-V_{\text{int}}(m)} \prod_{f\in m} \frac{|f|\alpha_{|f|}}{h_{\omega_2}(|f|)},$$

with $h_{\omega_2}(|\partial m|) \geqslant 1$. It follows that

$$\sup_{\omega\in\mathcal{N}} |p_m^{(\omega)}| \leqslant \frac{1}{\sqrt{1-\omega_1^{-1}}} \left(\frac{\omega_2 g(\omega_2)}{\omega_1 g(\omega_1)}\right)^{V_{\text{int}}(m)} \mathbb{P}\left(m \subset \mathbb{M}_{\mathbf{q}^{(\omega_2)}}\right),$$

so

$$\sum_{m\in\mathcal{H}} \sup_{\omega\in\mathcal{N}} |p_m^{(\omega)}| \leqslant \frac{1}{\sqrt{1-\omega_1^{-1}}}\mathbb{E}\left[\left(\frac{\omega_2 g(\omega_2)}{\omega_1 g(\omega_1)}\right)^{V_{\mathrm{int}}(B_1^\bullet(\mathbb{M}_{\mathbf{q}^{\omega_2}}))}\right]. \tag{7.6.6}$$

Therefore, it is sufficient to prove that for any $\omega_0 > 1$, we can find numbers $1 < \omega_1 < \omega_0 < \omega_2$ such that the last expectation is finite. This will follow from the next lemma.

Lemma 7.6.5. *Let $\omega_0 > 1$. We can find $1 < \omega_3 < \omega_0 < \omega_4 < +\infty$ and a constant $x > 1$ such that, for all $\omega \in [\omega_3, \omega_4]$, we have*

$$\mathbb{E}\left[x^{V_{\mathrm{int}}(B_1^\bullet(\mathbb{M}_{\mathbf{q}^{\omega}}))}\right] < +\infty.$$

First, let us explain how (7.6.6) follows from Lemma 7.6.5. We fix ω_3, ω_4 such that $1 < \omega_3 < \omega_0 < \omega_4 < +\infty$. Since $\omega \to \omega g(\omega)$ is continuous at ω_0, there are $\omega_1 \in (\omega_3, \omega_0)$ and $\omega_2 \in (\omega_0, \omega_4)$ such that

$$\frac{\omega_2 g(\omega_2)}{\omega_1 g(\omega_1)} < x.$$

Such ω_1 and ω_2 suit our needs.

Proof of Lemma 7.6.5. Sketch of proof:

- Step 1: pass from plane to half-plane (we can do it since peeling of plane = peeling of half-plane conditioned on an event of positive probability).

- Step 2: let τ be the number of peeling steps needed to finish exploring $B_1^\bullet$. Then $\mathbb{P}(\tau > k) \leqslant e^{-\alpha k}$ for k large enough, with α independent of $\omega \in [\omega_3, \omega_4]$.

- Step 3: let V_1 be the number of internal vertices created by one half-plane peeling step. We claim that there is $x > 1$ such that, for all $\omega \in [\omega_3, \omega_4]$, we have $\mathbb{E}[x^{V_1}] < e^{\alpha/2}$. Indeed, by summing over all peeling cases, we get

$$\mathbb{E}[x^{V_1} - 1] = 2\sum_{i\geqslant 0}\left(\frac{1}{\omega g(\omega)}\right)^{i+1}\sum_{|\partial m|=i}\left(x^{\#\mathrm{Vertices}(m)} - 1\right)\prod_{f\in m} q_{|f|}^{(\omega)}. \tag{7.6.7}$$

  The peeling cases where the peeled edge is glued to a new inner face do not contribute since then $x^{V_1} - 1 = 0$. Each term in this last sum is decreasing in ω and increasing in x, and equal to 0 for $x = 1$.

Therefore, to prove by dominated convergence that it is continuous on a neighbourhood of $(\omega_0, 1)$ (which implies our claim), it is enough to prove that, if we fix $1 < \omega_5 < \omega_0$, there is $x > 1$ such that $\mathbb{E}_{\omega_5}[x^{V_1}] < +\infty$. For this, we rewrite (7.6.7) by removing the -1 term, and we use the Euler formula to get rid of $x^{\#\text{Vertices}(m)} - 1$. We get

$$\begin{aligned}\mathbb{E}_{\omega_5}[x^{V_1} - 1] &\leqslant 2 \sum_{i \geqslant 0} \left(\frac{x}{\omega_5 g(\omega_5)} \right)^{i+1} W_i \left(\mathbf{q}^{(\omega_5, x)} \right) \\ &\leqslant 2 \sum_{i \geqslant 0} \left(\frac{x}{\omega_5 g(\omega_5)} \right)^{i+1} W_i^{\bullet} \left(\mathbf{q}^{(\omega_5, x)} \right),\end{aligned} \tag{7.6.8}$$

where $q_j^{(\omega,x)} = q_j^{(\omega)} x^{j-1}$, and $W_i^{\bullet}$ is the pointed partition function (i.e. biased by the total number of vertices). By definition of $\omega_{\mathbf{q}}$ in terms of the walk $\nu_{\mathbf{q}}$, we must have $\sum_{j \geqslant 1} q_j g(\omega_5)^j \omega_5^j \leqslant 1 < +\infty$, so the radius of convergence of (q_j) is at least $\omega_5 g(\omega_5)$, so it is larger than $g(\omega_5)$. This ensures that, for $x > 1$ small enough, the weight sequence $q_j^{(\omega_5,x)}$ is admissible. Moreover $g_{\mathbf{q}^{(\omega_5,x)}}$ is a continuous function of x in a neighbourhood of $x = 1$. By well-known combinatorial results [MM07, Bud15], we know that

$$W_i^{\bullet} \left(\mathbf{q}^{(\omega_5,x)} \right) = g^i_{\mathbf{q}^{(\omega_5,x)}} \times \frac{1}{4^i} \binom{2i}{i},$$

so the right-hand side of (7.6.8) is finite if and only if $g_{\mathbf{q}^{(\omega_5,x)}} < \frac{\omega_5 g(\omega_5)}{x}$. This is obviously true for $x = 1$, so this is true for some $x > 1$, which proves the claim.

- Step 4: $\mathbb{E}[x^{V_k}] = \mathbb{E}[x^{V_1}]^k$ because peeling steps are independent in the half-plane, so

$$\mathbb{E}[x^{V_\tau}] \leqslant \sum_{k \geqslant 1} \mathbb{P}\left(\tau > k\right) \mathbb{E}[x^{V_k}] \leqslant \sum_{k \geqslant 1} \mathbb{E}[x^{V_1}]^k < +\infty,$$

which finishes the proof.

$\square$

$\square$

Chapter 8

Open problems

We conclude this thesis by stating a few open problems that are in natural continuation of the work presented here.

8.1 Bijective problems

As explained in the introduction and in Chapter 4, a general goal is to prove the formulas arising from the hierarchies bijectively in full generality.

Open problem 1. *Find a unified bijection for the formulas arising from the KP and 2-Toda hierarchies.*

In order to do so, it would be reasonable to work with the "simplest" of those formulas, namely the Goulden–Jackson formula (1.2.2). Indeed, in the planar and one-faced cases, the bijection is always simpler for cubic maps than for general maps. Alternatively, one could try to find bijections for special cases (i.e. planar or one-faced) of the formulas of Chapter 5. So far, we only found bijections for the planar cases of the Goulden–Jackson and Carrell–Chapuy formulas. In some sense, the formulas of Chapter 5 (especially (1.3.4)) are more complex, and finding a bijective explanation could unveil some structural subtleties that were invisible in the simpler cases.

But the formulas arising from the hierarchies are not always recurrence formulas, they can be of a very different form, for instance the determinantal formula of [ACEH18, Proposition 4.1].

Finally, outside the KP/2-Toda world, there are other formulas on maps that are waiting for a bijective explanation. For instance, in [Tut74], Tutte proved a quadratic recurrence for properly coloured planar triangulations.

The coefficients that appear in the formula have a combinatorial interpretation.

Open problem 2. *Prove Tutte's recurrence for coloured triangulations.*

A corollary of this formula is a recurrence formula for planar triangulations without loops, which was recently proved bijectively by Abel Humbert (private communication).

8.2 Maps and the KP hierarchy

Outside the bijective world, the relation between maps and the hierarchies still has lots to offer.

Open problem 3. *Find more formulas for maps using the KP/2-Toda hierarchies.*

First, one can try to find a formula for constellations with prescribed face degrees. It would require a new approach, as in Chapter 5 the method we use has clear limitations (see Remark 5.3.2). Also, we started with a generating function that allows to control the cycle types of two permutations, but we always had to "trivialize" one of these two permutations for practical reasons. It would be interesting to derive formulas that control two cycle types (for instance, bipartite maps with prescribed degrees of black and white vertices).

A possible way to do so would maybe imply more equations than the simplest KP or 2-Toda equation. It would also involve more work on the structure of the generating function itself. As an illustration, all the formulas that were found before our work [GJ08, CC15, KZ15] use the KP equation as a blackbox, whereas our approach involves some manipulation of the tau-function itself. It is reasonable to expect that if there are more formulas to be found, they would also require manipulations on the tau-function.

Conversely, studying maps could give us a better understanding of the KP hierarchy. The first few[1] equations of the KP hierarchy are:

$$F_{3,1} = F_{2,2} + \frac{1}{2}F_{1,1}^2 + \frac{1}{12}F_{1^4},$$

$$F_{4,1} = F_{3,2} + F_{1,1}F_{2,1} + \frac{1}{6}F_{1^3,2},$$

$$F_{5,1} = F_{4,2} + \frac{1}{24}F_{1^4}F_{1^2} + F_{1^2}F_{3,1} + \frac{1}{480}F_{1^6} + \frac{1}{8}F_{1^2}^3 + \frac{1}{8}F_{1^2,2^2} + \frac{1}{2}F_{2,1}^2 + \frac{1}{8}F_{3,1^3}.$$

[1] in the sense of [MJD00].

We observe that all the equations above start with $F_{k,1} = F_{k-1,2} + \dots$, and all the coefficients are positive[2].

The bijections in Chapter 4 (and in particular Theorem 4.4.1) give an explanation of the "planar part" of the first KP equation. As noted in the introduction of Chapter 4 (paragraph **Discussion and related works**), Theorem 4.4.1 also explains the planar part of the second KP equation, and it can be applied to finding quadratic equations of the form $F_{k,1} = F_{k-1,2} + \dots$ for all k, where F is a well-chosen generating function of planar maps. On the other hand, thanks to the bijection for one-faced maps [Cha11], we have for all k a linear equation of the form $F_{k,1} = F_{k-1,2} + \dots$ where F is the generating function of one-faced maps. Every time, the coefficients that are found have a combinatorial interpretation.

Open problem 4. *Find, if it exists, a system S of equations of the form*

$$E_k := F_{k,1} = F_{k-1,2} + \dots$$

such that the RHS of E_k has positive coefficients, that is satisfied by all the solutions F of the KP hierarchy. Does S generate the KP hierarchy[3] ? Is it possible to have only quadratic equations in S ?

The existing bijections can help us conjecture the values of the coefficients in the equations E_k.

8.3 High genus maps

Geometric properties Now that the local limits of high genus maps have been studied, it is natural to ask for global geometric properties. The next big question in this domain is the diameter.

Conjecture 5. *Let g_n be such that $\frac{g_n}{n} \to \theta \in (0, 1/2)$, and let M_n be a uniform map of genus g_n with n edges. Then there exist two constants c and C depending only on θ such that*

$$c \log n < \mathrm{diam} M_n < C \log n$$

holds asymptotically almost surely.

[2] they are also rational, but it is clear from the construction of the KP hierarchy that all the equations must have rational coefficients.

[3] i.e. if F satisfies S then it is a solution of the KP hierarchy.

The idea behind this conjecture is the following: in the PSHT, the balls have exponential growth [Cur16], therefore the diameter should be logarithmic. Some evidence supports this conjecture: for instance, using the bounded ratio lemma, one can prove the lower bound of the conjecture. We end this paragraph with a (very ambitious) open question:

Open problem 6 (by Itai Benjamini)**.** *Let M_n defined as in the previous conjecture. Does there exist two constants $c, h > 0$ and a sequence of subgraphs G_n of M_n such that for all $n > 0$:*

- *G_n has at least cn edges,*
- *G_n is an h-expander ?*

Note that M_n itself cannot be an expander, since by our result, every finite planar pattern happens somewhere in M_n.

Scaling limits ? Another natural question, after proving the local convergence, is to investigate the scaling limit. Unfortunately, it is a general principle that a model whose diameter is not polynomial in the size cannot have a nice scaling limit. However, it is still possible to study some kind of "local scaling limit". Indeed, it was proved that the PSHTs have a scaling limit called the Hyperbolic Brownian Plane (HBP) [Bud18b].

Conjecture 7 (by Thomas Budzinski)**.** *Let $g_n = o(n)$ but $g_n \to \infty$ as $n \to \infty$, and let M_n be a uniform map of genus g_n with n edges. Fix a constant $a > 0$, and let $r_n = a\left(\frac{n}{g_n}\right)^{1/4}$. Call $B_{r_n}(M_n)$ the ball of radius r_n around the root of M_n. Then $B_{r_n}(M_n)$ converges[4] in distribution towards an instance of the HBP, whose parameter is determined by a.*

Let us make sense of this conjecture. First, a similar result exists in the planar case [CLG14]. Also, to obtain the HBP, the parameter λ of the PSHT has to go to λ_c as the scaling goes[5], therefore we must have $g_n = o(n)$. The parameter of the HBP is determined by how fast λ goes to λ_c. Since the HBP has a constant total curvature, so does $B_{r_n}(M_n)$. To ensure that, $B_{r_n}(M_n)$ must have a volume of order $\frac{n}{g_n}$. Since we have $g_n = o(n)$, the volume of $B_r(M_n)$ is of order $r^{1/4}$, so that explains why $r_n = a\left(\frac{n}{g_n}\right)^{1/4}$.

[4] when the graph distance is rescaled by r_n.

[5] it is proved in [Bud18b] that otherwise there is no scaling limit.

Extremal genus case. Something is missing in our study of high genus maps. What if $\frac{g_n}{n} \to \frac{1}{2}$? Let T_n be a uniform triangulation of genus g_n with $2n$ faces. In this case, the average of degree of the root of T_n goes to infinity, and therefore there is no local convergence and the sequence (T_n) is not even tight for the local topology. Still, we can ask some questions about the local behaviour around the root. For instance, in Chapter 6, the proof of planarity is "less tight" than the proof of one-endedness. Therefore, it could make sense to study the behaviour of

$$\mathrm{PR}_n = \max\{r | B_r(T_n) \text{ is planar}\}.$$

The relevant parameter seems to be $v_n = n + 2 - 2g_n$, the number of vertices in T_n.

Very large faces. After the results of Chapter 7, it is natural to ask what happens when the distribution of the degrees of the faces have an even heavier tail. Theorem 7.0.1 should still hold if we drop the condition $\sum_i i^2 \alpha_i < \infty$. Indeed, every part of the proof holds in this case except for the two holes argument, and the (expected) limiting object exists (see Proposition 7.1.3). A totally different regime (which is not covered by the IBPMs) would be the case $0 < \sum_i i\alpha_i < 1$. It means that there is a positive chance for the root face to be infinite, and therefore, in the limiting object, there would be infinite faces everywhere. We can expect that the local limit would look like a "thick tree", with an infinite number of infinite branches (probably like the local limit of unicellular maps [ACCR13]), that is "filled" with finite faces. To get started on this question, the following model should capture the essential features without too many parameters:

Open problem 8. *Fix $c > 0$, and let g_n be such that $\frac{g_n}{n} \to \theta > 0$. Let M_n be a uniform bipartite map of genus g_n with n quadrangles and one face of degree cn. Study the local convergence of the sequence M_n.*

Once this is understood, it would be interesting to obtain "local scaling limit" results for this model, it would be an analogue of some of the results that were obtained [BMR] for planar quadrangulations with a large boundary.

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